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Open Access Journal of Astronomy
ISSN :2996-6701

Multidimensional Interactive Cosmological Model: Chronological Evolution of Universe

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Review Article
Volume 2 Issue 2
Received Date: October 11, 2024
Published Date: October 30, 2024
DOI: 10.23880/oaja-16000135

Abstract

Cosmological studies incorporate a set of differential equations which can be shown as an autonomous system. In our article, interactions are chosen. Solution of the dynamical system is analysed. Chevallier- Polarski-Linder dark energy model is opted and the concerned stability is studied. Point of periodicity is found to grow at high redshift for low dimensionless density parameters of matter. As we chronologically proceed towards the present time, this point is shifted towards high values of dimensionless density due to matter. At present time even the point of periodicity is seen to sustain. Distinguishably no attractor or past time attractor is noticed to get formed.

Keywords: Autonomous Dynamical System; Dark Energy; Dark Matter; Interactions; Phase Space; PACS Numbers: 95.36.+x, 95.35.+d, 98.80.Cq

Abbreviations

ODEs: Ordinary Differential Equations; CPL: Chevallier-Polarski-Linder Parametrization.

Introduction

Most commonly studied cosmological models keep the system of autonomous ordinary differential equations as the governing equations. In general, we consider spatially flat (k = 0) universe and the corresponding matter contents are radiation with density ρr and equation of state $\omega = \frac{1}{3}$, perfect dust like fluid with density ρ_m with equation of state $\omega = 0$ and dark energy ρ_d equation of state $\rho_d = \omega_d \rho_d$.

We write the continuity equations and Friedmann equations and derive the autonomous system on them. This constructs a dynamical system.

A dynamical system can be any model which might range from a simple one like a single pendulum to a complex object like the human brain or the entire universe itself. It should mainly two distinct features, viz.,

- a space (state space or phase) and
- a mathematical rule describing the evolution of points in that space.

Two main types of dynamical systems are found: continuous which is standing on the evolution defined by a set of ordinary differential equations (ODEs). Second one is called time discrete dynamical systems which are defined by map or difference equations. When cosmology is concerned, Einstein's field equations for a homogeneous and isotropic space result in a system of ODEs and thus we are only interested in continuous dynamical systems.

Standard dynamical system is expressed as [1]

$$\dot{\vec{x}} = f(\vec{x}) \quad (1)$$

Where $\vec{x} = (x_1, x_2 \ldots, x_n) \in X$ is an element of the state space $X \in R^n, f : X \to X$ and over dot denotes differentiation with respect to some suitable time parameter.

For the autonomous system (1), if $f(x_0) = 0$ we treat $x = x_0$ as a critical point or a fixed point.

When x_0 be a fixed point and for every ϵ > 0 a δ(ϵ) can be found such that ψ(t) be a solution of (1). It satisfies the relation $\|\Psi(t_0) - x_0\| < \delta$, we conclude ψ(t) to exist for all $t \geq t_0$ and to satisfy $\|\Psi(t_0) - x_0\| < \delta$ for all $t \geq t_0$. In such a case x0 is said to be a stable fixed point. If the solutions approach the critical point for all nearly initial conditions, the fixed point is called asymptotically stable. x_0 is said to be hyperbolic if none of the eigen values of the

Jacobian matrix $J = \frac{\partial_{fi}}{\partial_{xj}}$ at x_0 have zero real part. If this condition is violated we tell the fixed point to be a nonhyperbolic one.

Eigen value studies show if all the eigen values are negative, we are obtained with an attractor of the system. All the trajectories will approach that fixed point. If, on the contrary, all the eigen values are positive, the fixed point is a past time attractor. If signs are different a saddle is found.

These mathematical views even have physical meanings in cosmological context. If we solve a set of simultaneous equations for cosmology. We observe if fixed points of (Ω_m, Ω_r) are located at (0,0), (0,1) and (1,0) respectively, attractor, past attractor and saddle attractor.

For more details we refer to the articles [2,3]. For anisotropic studies [4,5] can be reviewed. Based on the chronological epochs inflation→radiation→matter→cosmological constant/dark energy, pointing out it as a minimal dark energy model, we may establish links with dynamical systems. Cosmological model study in the Lotka-Volterra framework is [6].

We consider our model to be described by an n × n autonomous system of equations. This model is started with an inflationary period and then should correspond to the early time attractor in the corresponding dynamical system.

All eigen values of the Jacobian matrix at this point should be positive i.e, $\lambda_i = 0$ for i = 1, 2, ..., n. This ensures that all trajectories evolve away from this point.

An ideal model will show two saddle points ($\lambda_i > 0, \lambda_k < 0$ with j + k = n) which correspond to a radiation dominated and matter dominated universe respectively. As these epochs are saddle, some trajectories will be to these and be repelled from them as well. As they are saddle, the universe will evolve through this point. A late time attractor with $\lambda_i = 0$ for i = 1, 2, ..., n will be observed where the universe is undergoing an accelerated expansion.

Now, we have tried to explain the different phase evolutions of the universe with the help of some phase spaces. Again, we have considered the Chevallier-Polarski-Linder parametrization (hereafter CPL parametrization) of dark energy so that we can explain the late-time acceleration of the universe as well as some future predictions. We have considered the redshift z of the universe expansion in the CPL parametrization. So we will expect some variations in the phase spaces which can explain some new nature of the expansion of the universe.

It is well known from the literature that the CPL parametrization is one of the simplest models of dark energy. This has some interesting properties as discussed in the article [7]. So we have been motivated to carry forward our work with the CPL parametrization of dark energy. Using the CPL parametrization we are curious to study what kind of attractor or past-time attractor or equilibrium points we will get to explain the expansion of the universe with interacting components.

In this present article, we have obtained some basic equations in the first section. Next, we have constructed our interactive autonomous system using the coupling functions q_m and q_r. In this section, there are two subsections where we have obtained eigen values as well as the center equilibrium points after assuming $q_m = 0$ and $q_r = 0$. Later we have tried to draw the phase spaces with the CPL parametrization for some suitable amounts of taken parameters. In this last section there are two subsections. First subsection stands for the stability of the model and the other one contains some phase spaces and conclusion.

Mathematical Construction

We assume our universe to be filled up of baryonic matter, radiation and dark energy. The energy density is comprised of three distinct parts and the field equation looks like

$$H^2 = \frac{k}{3}\left(\rho_m + \rho_r + \rho_d\right) \qquad (2)$$

Along with this, the equations of continuity for matter, radiation and dark energy turn to be

$\dot{\rho}_m + 3H\rho_m = 0$ (as $\rho_m = 0$) (3)

$\dot{\rho}_r + 4H\rho_r = 0$ (as $\rho_r = \frac{1}{3}\rho_r$) and (4)

$$\dot{\rho}_d + 3H\rho_d\left\{\left(1+\omega_0\right) + \frac{\omega_1 z}{1+z}\right\} = 0 \qquad (5)$$

We derive an analogous equation with dimensionless density parameters as,

$1 = \Omega_m + \Omega_z + \Omega_d$

Where $\Omega_m = \frac{k\rho_m}{3H^2}, \Omega_z = \frac{k\rho_r}{3H^2}$ *and* $\Omega_d = \frac{k\rho_d}{3H^2}$ (6)

Differentiating the above first two expressions with respect to N = ln a(t), a(t) being the scale factor, we are led to derive the dynamical system as

$$\frac{d\Omega_m}{dN} = \Omega_m\left[3\left(\Omega_m - 1\right) + 4\Omega_r + 3\left(1 - \Omega_m - \Omega_r\right)\left\{\left(1+\omega_0\right) + \frac{\omega_1 z}{1+z}\right\}\right] \qquad (7)$$

and

$$\frac{d\Omega_r}{dN} = \Omega_r\left[4\left(\Omega_r - 1\right) + 4\Omega_m + 3\left(1 - \Omega_m - \Omega_r\right)\left\{\left(1+\omega_0\right) + \frac{\omega_1 z}{1+z}\right\}\right] \qquad (8)$$

Same kind of system is studied in the reference [8]. This reference has established that all the systems' equilibrium points are such as Ω_m or/and Ω_r do/does vanish. Hence non-coupled matter, radiation and dark energy cannot produce a homoclinic orbit which needs a center equilibrium point with $\Omega_i \neq 0$.

This does not support a ΛCDM cosmology. The same is found to be true for any extended theories of gravity with an effective dark energy [9]. For instance an f(R) theory in Palatini formalism can be written in the reference [10] as

$$\Omega_m = k\frac{k\rho_m}{3F\xi H^2}, \Omega_r = k\frac{k\rho_r}{3F\xi H^2} \text{ and } \Omega_f = \frac{FR - f}{6F\xi H^2} \qquad (9)$$

where R is the usual scalar curvature, $F = \frac{df}{dR}$ and

$\xi = \left[1 - \frac{3}{2}\frac{\frac{dF}{dR}\left(FR - 2f\right)}{F\left(\frac{dF}{dR} - F\right)}\right]^2$. With the help of these, we can rewrite the dynamical equations as

$$\frac{d\Omega_m}{dN} = \Omega\left[-3 + 3\Omega_m + 4\Omega_r + \left(1 - \Omega_m - \Omega_r\right)C\left(R\right)\right] \qquad (10)$$

and

with

$$\frac{d\Omega_r}{dN} = \Omega\left[-4 + 3\Omega_m + 4\Omega_r + \left(1 - \Omega_m - \Omega_r\right)C\left(R\right)\right] \qquad (11)$$

$$C\left(R\right) = -3\frac{\left(FR - 2f\right)\frac{dF}{dR}R}{FR - f\left(R\frac{dF}{dR} - F\right)}$$

Considering $C\left(R\right) = 3\left\{\left(1+\omega_0\right) + \frac{\omega_1 z}{1+z}\right\}$, we can construct the same dynamical system.

Interactive System

In this model, we consider a dark energy coupled to both radiation and matter with coupling functions Q_m and Q_r respectively. We construct

$q_m \frac{k}{3}\frac{Q_m}{H^3}$ and $q_r \frac{k}{3}\frac{Q_r}{H^3}$ (12)

Following the reference [11], we construct a dynamical system

$$\frac{d\Omega_m}{dN} = \Omega_m\left[-3 + 3\Omega_m + 4\Omega_r + 3\left(1 - \Omega_m - \Omega_r\right)\left\{\left(1+\omega_0\right) + \frac{\omega_1 z}{1+z}\right\}\right] + q_m \qquad (13)$$

and

$$\frac{d\Omega_r}{dN}=\Omega_r\left[-4+3\Omega_m+4\Omega_r+3\left(1-\Omega_m-\Omega_r\right)\left\{\left(1+\omega_0\right)+\frac{\omega_1 z}{1+z}\right\}\right]+q_r$$

(14)

Now, we have to choose some possible and suitable values for $\Omega_{m(eq)},\Omega_{r(eq)},\frac{dq_m}{d\Omega_m},\frac{dq_m}{d\Omega_r}$ *and* $\frac{dq_r}{d\Omega_m}$ such that some coupling functions allow homoclinic orbits around the center equilibrium point $\left(\Omega_{m(eq)},\Omega_{r(eq)}\right)$. Next, we will consider the special cases $q_m = 0$ and $q_r = 0$.

$q_m = 0$

We will try to find the central equilibrium points given by $\left(\Omega_{m(eq)},\Omega_{r(eq)}\right)\neq 0$ when $q_m = 0$, responsible for homoclinic orbits. Eigen values that have to be pure imaginary numbers take the form

$$\begin{aligned}2\lambda_\pm=\pm\Bigg[&9\Omega_m^4\frac{d\omega(z)^2}{d\Omega_m}+18\Omega_m^3\frac{d\omega(z)^2}{d\Omega_m}\left\{(\Omega r-1)\frac{d\omega(z)^2}{d\Omega_m}+\Omega_r\frac{d\omega(z)^2}{d\Omega_r}+\omega(z)\right\}+3\Omega_m^2\left\{\frac{d\omega(z)^2}{d\Omega_r}\left(3\Omega_r^2\frac{d\omega(z)^2}{d\Omega_m}-4\frac{dq_r}{d\Omega_m}\right)\right.\\&+2\frac{d\omega(z)^2}{d\Omega_m}\left(\frac{dq_r}{d\Omega_r}+6(\Omega_r-1)\Omega_r\frac{d\omega(z)}{d\Omega_r}-\Omega_r-1\right)+6\omega(z)\left((2\Omega r-1)\frac{d\omega(z)}{d\Omega_m}+\Omega_r\frac{d\omega(z)}{d\Omega_r}\right)+3(\Omega_r-1)^2\frac{d\omega(z)^2}{d\Omega_m}+3\omega(z)^2\Bigg\}\\&+2\Omega_m\left\{\frac{dq_r}{d\Omega_m}\left(-6\Omega_r\frac{d\omega(z)}{d\Omega_r}-6\omega(z)+6\frac{d\omega(z)}{d\Omega_r}+2\right)-3\Omega_r\frac{d\omega(z)}{d\Omega_r}\left(\frac{dq_r}{d\Omega_r}-6\Omega_r\omega(z)-3(\Omega_r-1)^2\frac{d\omega(z)}{d\Omega_m}+\Omega_r+3\omega(z)-1\right)\right.\\&\left.+3\left(\frac{dq_r}{d\Omega_r}+3\Omega_r\omega(z)-\Omega_r-1\right)\left((\Omega_r-1)\frac{d\omega(z)}{d\Omega_m}+\omega(z)\right)+9(\Omega_r-1)\Omega r^2\frac{d\omega(z)^2}{d\Omega_r}\right\}\\&+\left\{\frac{dq_r}{d\Omega_r}-3\Omega_r\omega(z)-3(\Omega_r-1)\Omega_r\frac{d\omega(z)}{d\Omega_r}+\Omega_r-1\right\}^2\Bigg]^{\frac{1}{2}}\\&-3\Omega_r\left\{\frac{d\omega(z)}{d\Omega_r}\right\}(\Omega_m+\Omega_r-1)+\Omega_m\frac{d\omega(z)}{d\Omega_m}+3\omega(z)-1+(6-9\Omega_m)\omega(z)-3(\Omega_m-1)\Omega_m\frac{d\omega(z)}{d\Omega_m}+\frac{dq_r}{d\Omega_r}-1\end{aligned}$$

(15)

Where $\omega(z)=\omega_0+\frac{\omega_1 z}{1+z}$. Now since for a center equilibrium point eigen values will be purely imaginary, there will exist only one center equilibrium point which is

$$\left(\Omega_{m(eq)},\Omega_{r(eq)}\right)=\left(1q_r+q_r\frac{1+z}{3\left(\omega_0 z+\omega_0+\omega_1\right)},q_r\right)$$

$$\frac{dq_r}{d\Omega_m}>(or<)\frac{\left\{q_r\frac{d\omega(z)}{d\Omega_m}+3\left(\omega_0+\frac{\omega_1}{z+1}\right)^2\right\}\left\{q_r^2\frac{d\omega(z)}{d\Omega_m}+3q_r\left(-q_r\frac{d\omega(z)}{d\Omega_m}+q_r\frac{d\omega(z)}{d\Omega_r}+\frac{d\omega(z)}{d\Omega_m}\right)\left(\omega_0+\frac{\omega_1}{z+1}\right)+9\left(\omega_0+\frac{\omega_1}{z+1}\right)^3\right\}}{3\left(\omega_0+\frac{\omega_1}{z+1}\right)^2\left\{q_r\frac{d\omega(z)}{d\Omega_r}+3\left(\omega_0+\frac{\omega_1}{z+1}\right)^2-\omega_0-\frac{\omega_1}{z+1}\right\}}$$

$$q_r\frac{d\omega(z)}{d\Omega_m}>(or<)\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\left\{1-3\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\right\}\ \textit{for}\ \left(\Omega_{m(\mathrm{eq})},\Omega_{r(\mathrm{eq})}\right)$$

$$\begin{aligned}\frac{dq_r}{d\Omega_r}=\Bigg[&1+3\omega_0+\frac{3\omega_1 z}{z+1}-q_r(z+1)\left\{q_r\frac{d\omega(z)}{d\Omega_m}\{3\omega_0(z+1)+3\omega_1 z-z-1\}\right.\\&\left.-3q_r\frac{d\omega(z)}{d\Omega_r}\{\omega_0+(\omega_0+\omega_1)z\}-3\frac{d\omega(z)}{d\Omega_m}\{\omega_0+(\omega_0+\omega_1)z\}\right\}\Bigg]\left[3\{\omega_0+(\omega_0+\omega_1)z\}^2\right]^{-1}\end{aligned}$$

(16)

and

$$\begin{aligned}2\lambda_\pm=\pm\Bigg[&2\Omega_r\left\{3\frac{d\omega(z)}{d\Omega_m}\left\{-\Omega_m\left(7\Omega_m+\frac{dq_m}{d\Omega_r}+2\frac{dq_m}{d\Omega_r}-3\omega(z)-3\right)-3(\Omega_m-2)(\Omega_m-1)^2\frac{d\omega(z)}{d\Omega_r}+2\frac{dq_m}{d\Omega_r}\right\}\right.\\&+3(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}\left\{-3(\omega(z)-1)+\frac{dq_m}{d\Omega_m}+6\omega(z)-5\right\}-3\left(3\frac{d\omega(z)^2}{d\Omega_r}+\omega(z)-4\right)\Omega_m\\&+9\frac{d\omega(z)^2}{d\Omega_r}(\Omega_m-1)\Omega_m^2+3\frac{dq_m}{d\Omega_m}\omega(z)-4\frac{dq_m}{d\Omega_m}-6\frac{dq_m}{d\Omega_r}\omega(z)+6\frac{dq_m}{d\Omega_r}+3\omega(z)-4\Bigg\}\\&+\Omega_m^2\{6\frac{d\omega(z)}{d\Omega_r}\left(-\Omega_m+\frac{dq_m}{d\Omega_m}+3\omega(z)-1\right)-6\omega_m\left(6(\Omega_m-2)(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}+(4-3\omega(z))\Omega_m\right.\end{aligned}$$

If

$$q_r\frac{d\omega(z)}{d\Omega_m}>(or<)\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\left\{1-3\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\right\}\ \textit{for}\ \left(\Omega_{m(\mathrm{eq})},\Omega_{r(\mathrm{eq})}\right)$$

(17)

$$\begin{aligned}2\lambda_\pm=\pm\Bigg[&2\Omega_r\left\{3\frac{d\omega(z)}{d\Omega_m}\left\{-\Omega_m\left(7\Omega_m+\frac{dq_m}{d\Omega_r}+2\frac{dq_m}{d\Omega_r}-3\omega(z)-3\right)-3(\Omega_m-2)(\Omega_m-1)^2\frac{d\omega(z)}{d\Omega_r}+2\frac{dq_m}{d\Omega_r}\right\}\right.\\&+3(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}\left\{-3(\omega(z)-1)+\frac{dq_m}{d\Omega_m}+6\omega(z)-5\right\}-3\left(3\frac{d\omega(z)^2}{d\Omega_r}+\omega(z)-4\right)\Omega_m\\&+9\frac{d\omega(z)^2}{d\Omega_r}(\Omega_m-1)\Omega_m^2+3\frac{dq_m}{d\Omega_m}\omega(z)-4\frac{dq_m}{d\Omega_m}-6\frac{dq_m}{d\Omega_r}\omega(z)+6\frac{dq_m}{d\Omega_r}+3\omega(z)-4\Bigg\}\\&+\Omega_m^2\{6\frac{d\omega(z)}{d\Omega_r}\left(-\Omega_m+\frac{dq_m}{d\Omega_m}+3\omega(z)-1\right)-6\omega_m\left(6(\Omega_m-2)(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}+(4-3\omega(z))\Omega_m\right.\\&\left.+2(\frac{dq_m}{d\Omega_m}+3\omega(z))\right)+9(\Omega_m-1)^2\frac{d\omega(z)^2}{d\Omega_r}+9\frac{d\omega(z)^2}{d\Omega_m}\Omega_m^2+15\omega(z)3\omega(z)-4)+16\Bigg\}\\&+\left\{-3\Omega_m\left(\frac{d\omega(z)}{d\Omega_m}(\Omega_m-1)+\omega(z)-1\right)+\frac{dq_m}{d\Omega_m}+1\right\}^2+6\Omega_r^3\left\{\frac{d\omega(z)}{d\Omega_r}\left(6\omega(z)-3(\Omega_m-2)\frac{d\omega(z)}{d\Omega_r}\right)\right.\\&\left.+\frac{d\omega(z)}{d\Omega_r}\left(3(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}+3\omega(z)-4\right)\right\}+9\frac{d\omega(z)^2}{d\Omega_r}\Omega_r^4\\&+\left\{-3\Omega_m\left(\frac{d\omega(z)}{d\Omega_m}(\Omega_m-1)+\omega(z)-1\right)+\frac{dq_m}{d\Omega_m}+1\right\}^2+6\Omega_r^3\left\{\frac{d\omega(z)}{d\Omega_r}\left(6\omega(z)-3(\Omega_m-2)\frac{d\omega(z)}{d\Omega_r}\right)\right.\\&\left.\left.+\frac{d\omega(z)}{d\Omega_r}\left(3(\Omega_m-1)\frac{d\omega(z)}{d\Omega_r}+3\omega(z)-4\right)\right\}+9\frac{d\omega(z)^2}{d\Omega_r}\Omega_r^4\right\}^{\frac{1}{2}}-3\Omega_m\left\{\frac{d\omega(z)}{d\Omega_m}(\Omega_r-1)+\frac{d\omega(z)}{d\Omega_r}+3\omega(z)-3\right\}\\&-3\frac{d\omega(z)}{d\Omega_m}\Omega_m^2+\frac{dq_m}{d\Omega_m}+3\Omega_r\left\{-\frac{d\omega(z)}{d\Omega_r}(\Omega_r-1)-3\omega(z)+4\right\}+6\omega(z)-7\end{aligned}$$

(18)

Center equilibrium point can be written as

$$\left(\Omega_{m(eq)},\Omega_{r(eq)}\right)=\left(-q_m,\frac{(z+1)q_m}{(3\omega_0-1)(z+1)+3\omega_1 z}+q_m+1\right)$$

where $\Omega_{m(eq)}>0,\Omega_{r(eq)}>0\,\text{and}\,\Omega_{m(eq)}+\Omega_{r(eq)}<1$ if

$$q_m<(3\omega_0-1)+\frac{3\omega_1 z}{z+1}$$

The Eigen values $\lambda_{\pm}$ will be purely imaginary numbers if

$$\frac{dq_m}{d\Omega_m}=\Bigg[3\frac{d\omega(z)}{d\Omega_r}q_m\left\{3(q_m+1)\left(\omega_0+\frac{\omega_1 z}{z+1}\right)-1\right\}+ \left(3\omega_0+\frac{3\omega_1 z}{z+1}-1\right)\left\{-3\frac{d\omega(z)}{d\Omega_m}q_m^2+9\left(\omega_0+\frac{\omega_1 z}{z+1}\right)^2-9\left(\omega_0+\frac{\omega_1 z}{z+1}+2\right)\right\}\Bigg]\left\{1-3\left(\omega_0+\frac{\omega_1 z}{z+1}\right)\right\}^{-2} \quad (19)$$

and

$$\frac{dq_m}{d\Omega_r}<(or>)\Bigg[\left[\left\{3\frac{d\omega(z)}{d\Omega_r}q_m+\left\{1-3\left(\omega_0+\frac{\omega_1 z}{z+1}\right)\right\}^2\right\}\left\{\left(3\omega_0+\frac{3\omega_1 z}{z+1}-1\right)\right. \left(\left\{1-3\left(\omega_0+\frac{\omega_1 z}{z+1}\right)\right\}^2-3\frac{d\omega(z)}{d\Omega_m}q_m^2\right)+3\frac{d\omega(z)}{d\Omega_r}q_m\left(3(q_m+1)\left(\omega_0+\frac{\omega_1 z}{z+1}\right)-1\right)\right\}\Bigg] \left[3\left\{1-3\left(\omega_0+\frac{\omega_1 z}{z+1}\right)\right\}^2\left\{\frac{d\omega(z)}{d\Omega_m}q_m+\left(\omega_0+\frac{\omega_1 z}{z+1}\right)\left(3\omega_0+\frac{3\omega_1 z}{z+1}-1\right)\right\}\right]^{-1}$$

(20)

when $q_m\frac{d\omega(z)}{d\Omega_m}>(or<)\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\left\{1-3\left(\omega_0+\frac{\omega_1 z}{1+z}\right)\right\}$ for $\left(\Omega_{m(eq)},\Omega_{r(eq)}\right)$.

Phase Space with CPL Parametrization

For the CPL parametrization $\omega(z)=\left\{\omega_0+\omega_1\frac{z}{1+z}\right\}$ [12], the best fit values of ω_0 and ω_1 with some data analysis, are $\omega_0=-0.97$ and $\omega_1=-0.57$ [13]. We have picked these values among tons of data analysis as we wanted such values of ω0 and ω1 which can signify the present time situation (phantom barrier), even when we do not consider the redshift. We have constructed a model inspired by the article [8],

$\omega_0=-0.97$ and $\omega_1=-0.57$,

$q_m=0$,

$q_r=\alpha\Omega_r\Omega_d$,

where α is a positive constant. Now, we are going to show the validity of stability of this model, using the equilibrium points we have got in the previous subsections.

Model Stability

We have equilibrium points for $\left(\Omega_m^{eq}\Omega_r^{eq}\right)\neq 0$, with the help of previous subsections as

$$\left(\Omega_m^{eq}\Omega_r^{eq}\right)=\left(1-\alpha\Omega_r(1-\Omega_r-\Omega_m)\left(\frac{1+z}{2.91z+4.62}+1\right),\alpha\Omega_r(1-\Omega_r-\Omega_m)\right)$$

Corresponding eigen values are

$$\begin{aligned}2\lambda_{\pm}=\pm1.71\Bigg[\Bigg\{&-2\Omega_m^{eq}\left(-\Omega_r^{eq}\frac{dz}{d\Omega_r}\left(\Omega_r^{eq}\left(z(5.99-1.17\alpha)-1.17\alpha-2\frac{dz}{d\Omega_m}+3.99\right)+0.59\alpha-0.59\alpha\Omega_m^{eq}\right.\right.\\
&+z\left(0.59\alpha-0.59\alpha\Omega_m^{eq}-3.29\right)+\left(\Omega_r^{eq}\right)^2\frac{dz}{2\Omega_m}+\frac{d}{2\Omega_m}-2.29\Big)-3.16\left\{\left(0.37\Omega_r^{eq}-0.37\right)\frac{dz}{d\Omega_m}+z+0.63\right\}\Big\{-0.5\alpha\\
&+0.5\alpha\Omega_m^{eq}+z\left(-0.5\alpha+0.5\alpha\Omega_m^{eq}+0.5\right)+\Omega_r^{eq}\left(\alpha(\alpha+2.81)z+1.955\right)+0.5\Big\}-\left(\Omega_r^{eq}-1\right)\left(\Omega_m^{eq}\right)^2\frac{dz}{d\Omega_r}^2\\
&+1.17\alpha\Omega_r^{eq}(z+1)\left(\left(\Omega_r^{eq}-1\right)\frac{dz}{d\Omega_r}+3.29z+2.29\right)\Big)\Big)+\Big(\Omega_r^{eq}\Big(z(3.29-1.17\alpha)-1.17\alpha-\frac{dz}{d\Omega_r}+2.29\Big)\\
&+0.59\alpha-0.59\alpha\Omega_m^{eq}+z\left(0.59\alpha-0.59\alpha\Omega_m^{eq}-0.59\right)+\left(\Omega_r^{eq}\right)^2\frac{dz}{d\Omega_r}0.59\Big)^2+\left(\Omega_m^{eq}\right)^4\frac{dz}{d\Omega_m}^2\\
&+2\left(\Omega_m^{eq}\right)^3\frac{dz}{d\Omega_m}\left(\left(1\Omega_r^{eq}-1\right)\frac{dz}{d\Omega_m}+\Omega_r^{eq}\frac{dz}{d\Omega_r}+2.7z+1.7\right)+\left(\Omega_m^{eq}\right)^2\Big(4\frac{dz}{d\Omega_m}\left(0.29\alpha(z+1)\left(\Omega_m^{eq}+\Omega_r^{eq}-1\right)\right.\\
&+0.29\alpha\Omega_r^{eq}(z+1)+0.29\Omega_r^{eq}(z+1)+\left(\Omega_r^{eq}-1\right)\Omega_r^{eq}\frac{dz}{d\Omega_r}+0.29(z+1)\Big)+10.81(z+0.63)\Big\{\left(\Omega_r^{eq}-0.5\right)\frac{dz}{d\Omega_m}\\
&+0.5\Omega_r^{eq}\frac{dz}{d\Omega_r}\Big\}+\Omega_r^{eq}\frac{dz}{d\Omega_r}\left(\Omega_r^{eq}\frac{dz}{d\Omega_r}-2.34\alpha(z+1)\right)+\left(\Omega_r^{eq}-1\right)^2\frac{dz}{d\Omega_m}^2+7.3(z+0.62)^2\Big)\Bigg\}(z+1)^{-2}\Bigg]^{\frac{1}{2}}\\
&-\alpha\left(\Omega_m^{eq}+\Omega_r^{eq}-1\right)+\frac{1.71\Omega_r^{eq}\left(\Omega_m^{eq}\frac{dz}{d\Omega_m}+\Omega_m^{eq}\frac{dz}{d\Omega_r}+\Omega_r^{eq}\frac{dz}{d\Omega_r}+9.86z-\frac{dz}{d\Omega_r}+6.86\right)}{z+1}+\frac{1.71\left(\Omega_m^{eq}-1\right)\Omega_m^{eq}\frac{dz}{d\Omega_m}}{z+1}\\
&+\left(6-9\Omega_m^{eq}\right)\left(-\frac{0.57z}{z+1}-0.97\right)-\alpha\Omega_r^{eq}-1\end{aligned}$$

Condition for above eigen values will be purely imaginary:

$$\alpha=\left\{-6.82-10.24z+1.71(\Omega_m)^2\frac{dz}{d\Omega_m}+\Omega_m\left(8.73+13.86z-1.71\frac{dz}{d\Omega_m}+1.71\Omega_r\frac{dz}{d\Omega_m}+1.71\Omega_r\frac{dz}{d\Omega_r}\right)+\Omega_r\left(11.73+16.86z-1.71\frac{dz}{d\Omega_r}+1.71\Omega_r\frac{dz}{d\Omega_r}\right)\right\}\left\{(-1+\Omega_m+2\Omega_r)(1+z)\right\}^{-1}$$

this is the restriction for α we have used in our model. Again, the equilibrium points (Ωeq, Ωeq) agree an homoclinic orbit is on the relation

$$\Omega_r^{eq}=\left(1-\Omega_m^{eq}\right)\left(\frac{1+z}{2.91z+4.62}+1\right)^{-1}$$

Phase Space Analysis and Conclusion

Using the previous constructed dynamical system, we have drawn phase portraits to observe that if our model is able to explain the evolution of the universe or not. So, next we have tried to explain them by analysis.

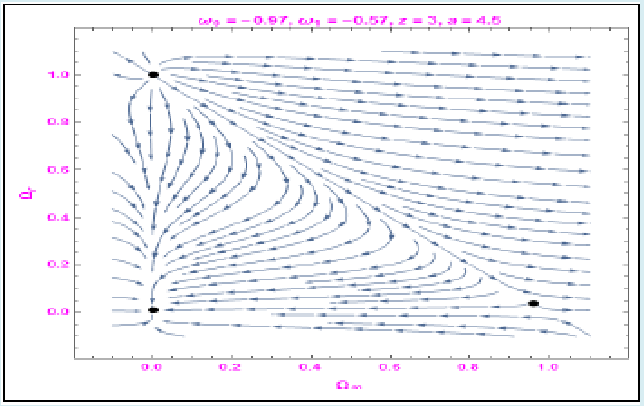

Figure 1: Is the phase space for $\omega_0 = -0.97$, $\omega_1 = -0.57$, α = 4.5 and z = 3.

We have drawn a phase portrait (Figure 1), when α = 4.5 and z = 3. We can easily observe that actually it contains three points- two repeler points and one attractor i.e., two unstable points and one stable point.

First, let's talk about the rappeler points. We have the first repellent at $\Omega_r \approx 1, \Omega_m \approx 0$. Clearly this point explains the inflation stage of the universe as well as the radiation dominated era i.e., the stage of the universe right after the Big Bang when there is only radiation and no existence of dark matter, dark energy. At this stage, the only cause of expansion of the universe is radiation. We can observe from the above phase portrait that every contour line is gone away from this point implies, this unstable point signifies an unstable universe and after that universe evolves. Again we have a second rappeler point at $\Omega_m \approx 1, \Omega_r \approx 0$. This signifies the universe is completely dominated by dark matter. This point also signifies the unstable universe and after that universe evolves. Now, let's talk about the attractor point.

We have got this point at $\Omega_d \approx 1$ i.e, the universe is fully dominated by dark energy. This attractor as well as stable point signify a stable universe. Now, we have constructed the whole dynamical system after considering the CPL parametrization of dark energy and we have taken the coupled function q_r in our system. This coupled function qr represents the interaction between radiation and dark energy. After all these facts, we have an attractor which signifies such an incident. These kind of phase portraits are almost independent of redshift z i.e., during the plots of these phase portraits we have always the same three points at almost the same location for low and high redshifts. So we can conclude that if this kind of interaction between radiation and dark energy has taken place, our dynamical system does not support the future deceleration of the universe. Again since we have got the full dominance of the universe by dark energy- this rises to the fact that in the future the universe will undergo the cosmological singularity phenomena Big Rip. Now, we have considered the value of $\omega_0 = -0.97$, $\omega_1 = -0.57$ for our dark energy parametrization and if we will consider the present time redshift z(=0) of the universe then we can also mention the date of this Big Rip w.r.t present time. The famous physicist Robert R. Caldwell gave a general Big Rip scenario. According to this scenario for our model there will be a Big Rip after 38 billion years from now.

Surprisingly we have identified some different behaviour of phase portraits when α > 5.4. These types of phase spaces are not independent of redshift z. These phase spaces are analysed below.

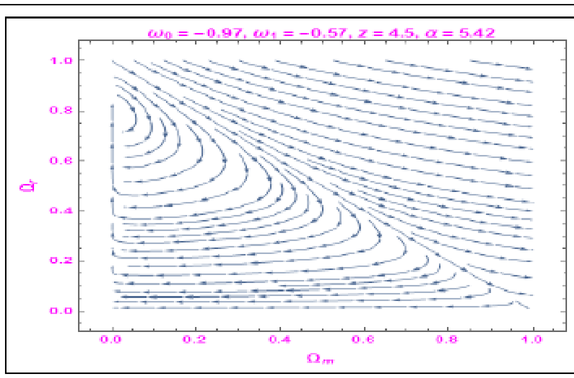

Figure: 2.1

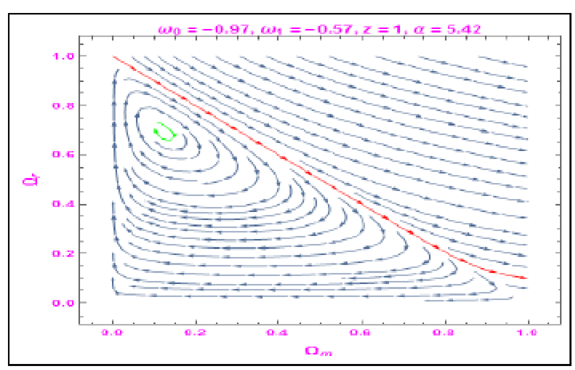

Figure: 2.2

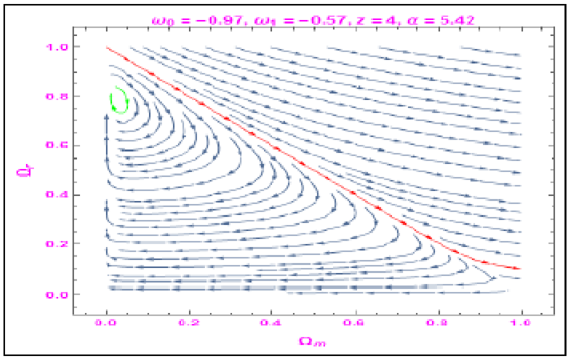

Figure: 2.3

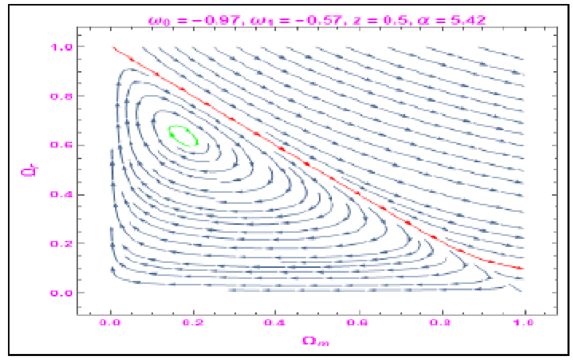

Figure: 2.4

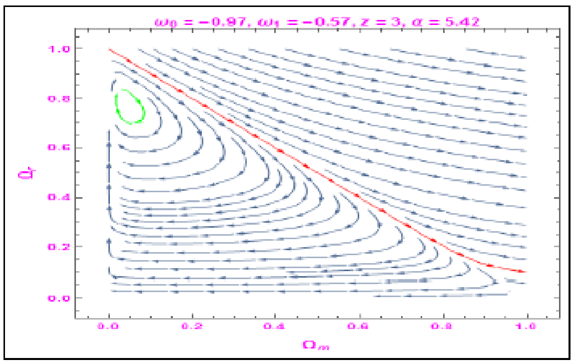

Figure: 2.5

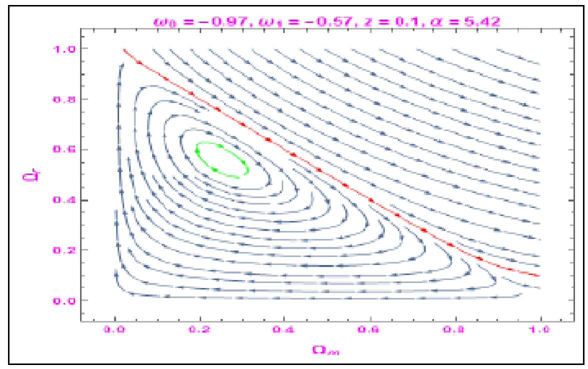

Figure: 2.6

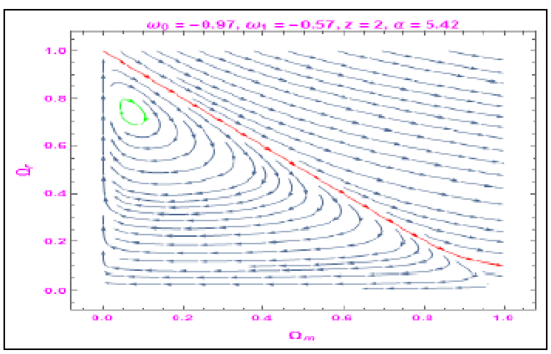

Figure: 2.7

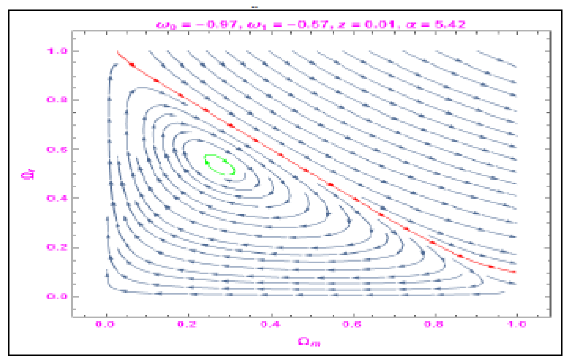

Figure: 2.8

Figure 2.1 to 2.8: Are the phase spaces for different redshifts at $\omega_0 = -0.97$, $\omega_1 = -0.57$ and α = 5.42.

We have plotted some phase portraits for different values of redshift z when α = 5.42. From the above phase spaces, we can observe the periodic behaviour of our coupled model for z < 4.5 (Figure 2.1). We have faced such kind of phenomena as for our system when z < 4.5 we did not get any purely imaginary eigen values and there is no any center equilibrium point. Now we can see that for redshift z = 0.01 to z = 4, we have seven phase spaces (Figure 2.2-2.8) and each of this has a common pattern of contour lines. Every phase space has a red contour line. The lower portion of that red line stands for positive densities and the upper portion of this red line represents negative densities which are not our interest though. In the area of positive densities, a circular green contour line is appeared for each of the seven phase spaces. This green circular contour line represents the point of periodicity of our coupled model. We have noticed this point of periodicity moves downwards when the redshift z decreases from 4 to 0.01. This phenomena can be explained by the fact that when we consider the past accelerated situation of the universe to present time acceleration of the universe, value of the density parameter of dark matter will remain structured as we did not take any interaction of dark matter and dark energy i.e. $q_m = 0$. Again, since we have assumed the interaction between radiation and dark energy, there is an abnormal variation in the values in Ωr. Since we have taken the coupled function $q_r = \alpha\Omega_r\Omega_d$, the interaction between radiation and dark energy become stronger. So, when we will defer redshift from z = 4 to z = 0.01 i.e., from past time situation of the universe to present era of the universe, we will get more accurate value of Ωm but value of Ω_r defers. At redshift z = 0.01, we have $\Omega_m = -0.26$. Now, due to the high value of α the interaction has taken place between radiation and dark energy, becoming very high. For this purpose we have got some abnormal values of Ω_r.

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