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Appendix A: Derivation of Expression for the Area of Bow-Shaped Segments

The general bow-shape of interest is illustrated in Figure 2. An expression for the shaded area can be obtained from integration of the equation for ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{A.1}$$

The shaded area is expressed as

$$A=\int_{-b}^{y} x\,dy \tag{A.2}$$

where *x* is given by Eq. (A.1) as

$$x=\frac{a}{b}\sqrt{b^2-y^2} \tag{A.3}$$

Substituting Eq. (A.3) into Eq. (A.2) gives

$$A=\int_{-b}^{y}\frac{a}{b}\sqrt{b^2-y^2}\,dy \tag{A.4}$$

which is integrated to yield

$$A=\frac{a}{b}\left[\frac{\pi}{2}b^2+y\sqrt{b^2-y^2}+b^2\arcsin\left(\frac{y}{b}\right)\right] \tag{A.5}$$

For a given segment height *h*, *y=h-b*. Eq. (A.5) becomes

$$A=\frac{a}{b}\left[\frac{\pi}{2}b^2+(h-b)\sqrt{b^2-(h-b)^2}+b^2\arcsin\left(\frac{h-b}{b}\right)\right] \tag{A.6}$$

For a segment of the circular shape, *a=b=R*. Eq. (A.5) becomes

$$A=\frac{\pi}{2}R^2+y\sqrt{R^2-y^2}+R^2\arcsin\left(\frac{y}{R}\right) \tag{A.7}$$

For a given segment height *h*, *y=h-R*. Eq. (A.7) becomes

$$A=\frac{\pi}{2}R^2+(h-R)\sqrt{R^2-(h-R)^2}+R^2\arcsin\left(\frac{h-R}{R}\right) \tag{A.8}$$

Appendix B: Derivation of Expression of Frictional Pressure Gradient for Laminar Flow through a Channel of Bow-Shaped Cross Section

The equation of the ellipse shown in **Figure 6** is:

$$\frac{x^2}{a^2} + \frac{(y-b)^2}{b^2} = 1 \quad \text{(B.1)}$$

Therefore,

$$x = \frac{a}{b}\sqrt{b^2 - (y-b)^2} \quad \text{(B.2)}$$

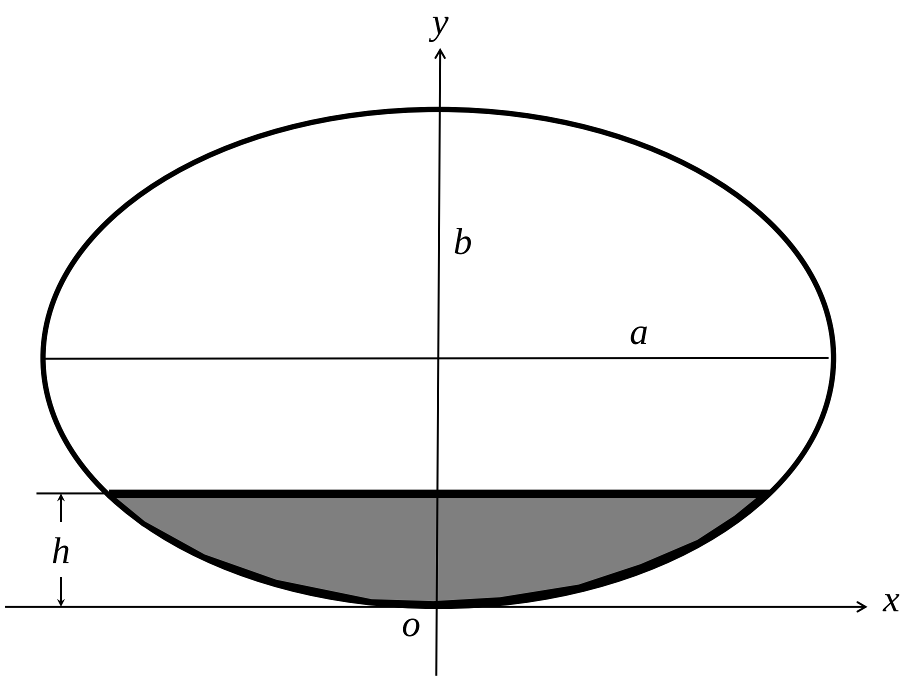

Figure 6: A general bow-shaped area as the bottom segment of an ellipse

The width of the shade area 2*x* is height-dependent:

$$2x = \frac{2a}{b}\sqrt{b^2 - (y-b)^2} \quad \text{(B.3)}$$

The left side of **Figure 7** illustrates profiles of flow velocity and shear stress in the flowing direction *z*. The right side of the figure shows a sketch of viscous forces acting on a fluid element with a length of Δz, a thickness of Δy, and a width of 2*x* (not shown).

The force F_1 applied by the fluid pressure at Point 1 is given by

$$F_1 = 2x\Delta yp \quad \text{(B.4)}$$

where *p* is pressure. Likewise, the force F_2 applied by the fluid pressure at Point 2 is given by

$$F_2 = 2x\Delta y\left(p - \frac{dp}{dz}\Delta z\right) \quad \text{(B.5)}$$

The frictional force exerted by the adjacent layer of fluid below the fluid element of interest is given by

$$F_3 = 2x\Delta z\tau \quad \text{(B.6)}$$

where τ is the shear stress. Similarly, the frictional force exerted by the adjacent layer of fluid above the fluid element of interest is given by

$$F_4 = 2x\Delta z\left(\tau + \frac{d\tau}{dy}\Delta y\right) \tag{B.7}$$

Under steady flow conditions, the sum of all these forces must be equal to zero:

$$F_1 - F_2 + F_3 - F_4 = 0 \tag{B.8}$$

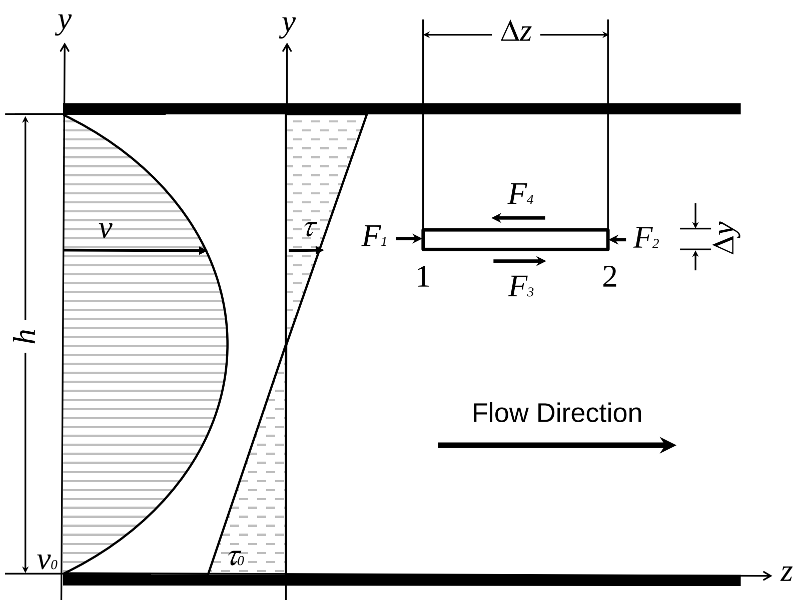

Figure 7: Velocity, shear stress, and viscous forces acting on a fluid element

Substituting Eqs. (B.4) through (B.7) into Eq. (B.8) and dividing the resultant equation through by ($2x\Delta y\Delta z$) yield

$$\frac{dp}{dz} - \frac{d\tau}{dy} = 0 \tag{B.9}$$

Separation of variables gives:

$$d\tau = \frac{dp}{dz}dy \tag{B.10}$$

which is integrated to get:

$$\tau = \frac{dp}{dz}y + \tau_0 \tag{B.11}$$

where τ_0 is the shear stress at $y=0$. The shear rate γ is given by

$$\gamma = -\frac{dv}{dy} \tag{B.12}$$

Thus for Newtonian fluids we obtain

$$\tau = \mu\gamma = -\mu\frac{dv}{dy} \tag{B.13}$$

Substituting Eq. (B.11) into Eq. (B.13) gives

$$\frac{dp}{dz}y + \tau_0 = -\mu\frac{dv}{dy} \tag{B.14}$$

Separating variables gives

$$dv = \frac{1}{\mu}\left(-\frac{dp}{dz}y - \tau_0\right)dy \tag{B.15}$$

which is integrated to give

$$v = -\frac{y^2}{2\mu}\frac{dp}{dz} - \frac{\tau_0}{\mu}y + v_0 \tag{B.16}$$

where v_0 is fluid velocity at $y=0$. Since the fluid wets the slot walls, the velocity v_0 is zero for $y=0$ and $y=h$. Applying these boundary conditions to Eq. (B.16) yields

$$0 = -0 - 0 + v_0 \tag{B.17}$$

and

$$0 = -\frac{h^2}{2\mu}\frac{dp}{dz} - \frac{\tau_0}{\mu}h + v_0 \tag{B.18}$$

which give

$$v_0 = 0 \tag{B.19}$$

and

$$\tau_0 = -\frac{h}{2}\frac{dp}{dz}. \tag{B.20}$$

Substituting these two expressions to Eq. (B.16) yields

$$v = \frac{1}{2\mu}\frac{dp}{dz}\left(hy - y^2\right) \tag{B.21}$$

The total fluid flow rate is expressed as

$$Q = \int_0^h v w_{eq} dy \tag{B.22}$$

where the equivalent width is defined as

$$w_{eq} = \frac{A}{h} = \frac{a}{hb}\left[\frac{\pi}{2}b^2 + (h-b)\sqrt{b^2 - (h-b)^2} + b^2 \arcsin\left(\frac{h-b}{b}\right)\right] \tag{B.23}$$

Substituting Eqs. (B.21) and (B.23) into Eq. (B.22) gives

$$Q = \int_0^h \frac{w_{eq}}{2\mu} \frac{dp}{dz} \left(hy - y^2 \right) dy \qquad \text{(B.24)}$$

which is integrated to give

$$Q = \frac{w_{eq} h^3}{12\mu} \frac{dp}{dz} \qquad \text{(B.25)}$$

which yields

$$\frac{dp}{dz} = \frac{12\mu Q}{w_{eq} h^3} \qquad \text{(B.26)}$$

Expressing the flow rate in terms of the mean flow velocity $v_{av} = \frac{Q}{w_{eq} h}$ gives

$$\frac{dp}{dz} = \frac{12\mu v_{av}}{h^2} \qquad \text{(B.27)}$$

Converting from consistent units to U.S. oilfield units of psi/ft, cp, ft/s, and inch, we obtain

$$\frac{dp}{dz} = \frac{\mu v_{av}}{1{,}000 h^2} \qquad \text{(B.28)}$$

Eq. (B.28) is valid for Newtonian fluids only. For non-Newtonian fluids, the Newtonian viscosity μ is replaced by the apparent Newtonian viscosity μ_a (Guo and Liu, 2011). For Bingham plastic fluids, the apparent Newtonian viscosity is expressed as

$$\mu_a = \mu_p + \frac{5\tau_y h}{v_{av}} \qquad \text{(B.29)}$$

where μ_a is apparent viscosity in cp, μ_p is plastic viscosity in cp, and τ_y is yield point in lbf/100ft^2.

For Power Law fluids, the apparent Newtonian viscosity is expressed as

$$\mu_a = \frac{K h^{1-n}}{144 v_{av}^{1-n}} \left(\frac{2 + 1/n}{0.0208} \right)^n \qquad \text{(B.30)}$$

where K is consistency index in cp equivalent and n is flow behavior index.