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Canonical Quantization of Yang-Mills Theory in Curved Space-time

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Abstract

The framework of canonical quantization of quantum field is given by Lagrangian (not Lagrangian density) formulation to avoid the problem of transfer physical quantities from 4–dimensional space–time to 3–dimensional position space (3–dimensional hypersurface with normal vector dt).

Keywords: Canonical quantization; Yang-Mills theory in curved space-time

Introduction

The canonical quantization of quantum field in curved space–time [1-3] is a good question when we want to develop the quantum field theory from flat space-time to curved one. There are well defined Klein–Gordon [3-5] and Dirac theory [6] in curved space–time. The Lagrangian density formulation of quantum field in curved space–time have to deal with the problem of transfer 4–dimensional space–time to 3–dimensional position space physical quantities when we use the Legendre transformation from Lagrangian density to Hamiltonian density, where 3–dimensional position space means 3–dimensional hypersurface with normal vector dt. There are lots of investigations of quantum field theory in curved space–times from the path integral method [5] and WKB approach [7], etc. The formal theory from sheaf quantization to path integral quantization hints us that we can just deal with 4–dimensional space–time physical quantities in the formulation of Lagrangian and Hamiltonian [8], which divides the space–time into 2 parts, position space and time. And we have to note that this paper avoids dealing with Lagrangian density, the Lagrangian density formulation divides the space–time to 4 parts, positions x, y, z and time t, this is the origination of intricate for some traditional canonical quantization scheme. Based on the Lagrangian and Hamiltonian formulation in local coordinates, the canonical quantization of Klein–Gordon, Dirac, Maxwell and Yang–Mills theories in local coordinates are self–consistently shown in this note. This formulation of canonical quantization of quantum field in curved space–time might enhance our understanding of quantum cosmology, Unruh effect and black hole entropy.

The second section gives us a formal framework of canonical quantization of quantum field in curved space–time. The third section discusses the canonical quantization of Klein–Gordon theory in curved space–time. The fourth section deals with canonical quantization of Dirac theory in curved space–time. The fifth section shows the canonical quantization of Maxwell theory in curved space–time. The sixth section talks about canonical quantization of Yang–Mills theory, especially the ghost fields, in curved space–time. We end this paper by seventh section of discussion.

Framework of Canonical Quantization of Quantum Field

Hamiltonian and Lagrangian

The measurable quantities in quantum theory have the formalism

$$\left\langle \psi' \left| \hat{0} \right| \psi \right\rangle, \quad (2.1)$$

where $\hat{0}$ is the operator of the corresponding measurable quantity, $\left|\psi\right\rangle$ is quantum state in Hilbert space. For any mechanical quantity, the average value of the measurable quantity is real

$$\left\langle \psi \left| \hat{0} \right| \psi \right\rangle \in R, \quad (2.2)$$

the mechanical quantity corresponded operator is Hermitian

$$\hat{0}\dagger = \hat{0}. \quad (2.3)$$

Further, the Hamiltonian as one of mechanical operator, has the formula

$$\hat{H} \approx aa\dagger,$$

such that

$$\left\langle \psi \left| \hat{H} \right| \psi \right\rangle = E_{total} \geq 0, E_{total} \in R. \quad (2.5)$$

In quantum field theory (second quantized), the state vector of quantum field theory $\left|\psi\right\rangle$ in huge–Hilbert space is evolved by Schrödinger equation

$$i\frac{\partial\left|\psi\right\rangle}{\partial t} = \hat{H}\left|\psi\right\rangle \quad (2.6)$$

where the Hamiltonian $\hat{H}$ is function of canonical position φ_κ and momentum π_k

$$\hat{H} = \hat{H}(\varphi_k, \pi_k), \quad (2.7)$$

the Hamiltonian $\hat{H}$ is mechanical quantity operator of energy. The position space V is a 3–dimensional hypersuface of 4–dimensional space–time manifold M with normal vector dt, θ_v is volume density of manifold M''.

$$\theta_v = \det(\theta^a_\mu), \quad (2.9)$$

$g_{\mu\nu}$ are components of metric tensor

$$d^2 s = -g_{\mu\nu} dx^\mu \otimes dx^\nu, \quad (2.10)$$

of manifold M with signature $(-,+,+,+)$. The canonical position φ_k and momentum π_k are q–number valued field operators. The Hamiltonian and Lagrangian are related by Legendre transformation

(2.11)

There is Euler–Lagrange equation related with the Lagrangian $\hat{L}$

$$\partial_\mu\left(\frac{\partial\hat{L}}{\partial(\partial_\mu\varphi_\kappa)}\right) - \frac{\partial\hat{L}}{\partial\varphi_\kappa} = 0. \quad (2.12)$$

For bosonic field, there is canonical commutation relation between canonical position and momentum

$$[\varphi_K(\vec{x},t), \pi_\kappa(\vec{x},t)] = ik\delta_{\kappa\kappa'}, \quad (2.13)$$

for fermionic field, there is canonical anti–commutation relation

$$\{\varphi_k(\vec{x},t), \pi_{\kappa'}(\vec{x},t)\} = ik\delta_{\kappa\kappa'}, \quad (2.14)$$

where k is the number of components of bosonic or fermionic fields. For example, the fermionic Dirac spinnor ψ in 4–dimensional space–time has 4 components, then

$$k = 4. \quad (2.15)$$

Schrödinger and Interaction Pictures

In Schrödinger picture, the Hamiltonian can be chosen as time free operator, and the canonical position φ_κ and momentum π_κ can be chosen as function of position

$$\varphi_k = \varphi_k(\vec{x}), \pi_\kappa = \pi_\kappa(\vec{x}).. \quad (2.16)$$

The time evolution matrix in Schrödinger picture can be derived from Schrödinger equation

$$S(t,t_0) = e^{-i\hat{H}(t-t_0)}. \quad (2.17)$$

For interaction quantum field, the Hamiltonian can be written

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \quad (2.18)$$

where $\hat{H}_0$ is free Hamiltonian gathered by kinematic terms, the $\hat{H}_{\text{int}}$ is interaction Hamiltonian and gathered by

interaction terms of Lagrangian

$$\hat{H}_{\text{int}} = -\hat{L}_{\text{int}}. \quad (2.19)$$

The effects of kinematic terms and interaction terms can be divided by interaction picture, and the wave function in interaction picture is defined as follow

$$\left|\Psi(t)\right\rangle = e^{-i\hat{H}_0 t}\left|\psi_{\acute{E}}(t_0)\right\rangle. \quad (2.20)$$

Then, the evolution equations of operator and wave function in interaction picture are

$$\frac{d\hat{O}}{dt} = i[\hat{H}_0, \hat{O}] \quad (2.21)$$

$$i\frac{\partial|\Psi_I\rangle}{\partial t} = \hat{H}_{\acute{E}}(t)|\Psi_I\rangle, \quad (2.22)$$

where

$$\hat{H}_{\acute{E}}(t) = e^{i\hat{H}_0 t}\hat{H}_{\text{int}}(t)e^{-i\hat{H}_0 t}, \quad (2.23a)$$

$$\hat{H}_0 = e^{i\hat{H}_0 t}\hat{H}_0 e^{-i\hat{H}_0 t}, \quad (2.24b)$$

with

$$\hat{H}_I = -\hat{L}_I. \quad (2.25)$$

The time evolution matrix in interaction picture $U(t,t_0)$ is

$$|\varnothing_{\acute{E}}(t)\rangle = U(t,t_0)|\varnothing_{\acute{E}}(t_0)\rangle, \quad (2.26)$$

and

$$i\frac{\partial}{\partial t}U(t,t_0) = \hat{H}_{\acute{E}}(t)U(t,t_0). \quad (2.27)$$

The time evolution matrix U satisfy the relations as follow

$$U(t_0,t_0) = I, \quad (2.28)$$

$$U(t_1,t_0) = (t_1,t)U(t,t_0), \quad (2.29)$$

$$U(t,t_0)^{-1} = U(t,t_0)\dagger = U(t_0,t). \quad (2.30)$$

The Dyson series expansion of evolution matrix U is [9]

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\hat{H}_I(t')\right]\right\} = 1 + (-i)\int_{t_0}^{t} dt_1\hat{H}_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 dt_2 T\left\{\hat{H}_I(t_1)\hat{H}_I(t_2)\right\} + \ldots \quad (2.31)$$

where T is time ordered operator. The relation between S and U time evolution matrix is

$$U(t,t_0) = e^{iH_0(t-t_0)}S(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \quad (2.32)$$

Canonical Quantization of Klein–Gordon Theory in Curved Space–time

Klein–Gordon equation describes the relativistic scalar particles (spin–0). The Lagrangian of Klein–Gordon theory in 4–dimensional curved space–time M is

$$\hat{L}_{KG} = \int d^3x \frac{\theta_v}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2), \quad (3.1)$$

where $g^{\mu\nu}$ are components of inverse metric. The indices μ, ν and a, b here are coordinate and orthogonal frame indices

$$\mu,\nu = 0,1,2,3 = t,x,y,z, \quad (3.2a)$$

$$a,b = 0,1,2,3. \quad (3.2b)$$

The Lagrangian (3.1) hints that the energy–momentum–mass of Einstein relation in curved space–time should be

$$g^{\mu\nu}p_\mu p_\nu = m^2, \quad (3.3)$$

where the p_μ is 4–momentum

$$p_\mu = (p_t, p_x, p_y, p_z) = (p_t, \vec{p}) = (p_t, p_q). \quad (3.4)$$

with

$$q = 1,2,3 = x,y,z. \quad (3.5)$$

The canonical positions of Klein–Gordon theory in curved space–time are

$$\varphi_k = \{\phi, \theta_a^\mu\}, \quad (3.6)$$

the corresponding canonical momentums are

$$\pi_k = \left\{\int d^3x\theta_v\, g^{\mu t}\partial_\mu\phi, 0\right\} \quad (3.7)$$

In quantum field theory, the scalar field is spanned by annihilation and creation operators locally

$$\phi(x) = \phi^- + \phi^+ = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{-i\vec{p}.\vec{x}} + a_{\vec{p}}^\dagger e^{ip.x}\right) \quad (3.8)$$

and the orthogonal frame coefficients still with the usual c–number valued

$$\theta_a^\mu, \theta_\mu^a \in \mathbb{R}. \quad (3.9)$$

To satisfy the canonical commutation relation between canonical position and momentum of Klein–Gordon theory

$$[\varphi_k(\vec{x}), \pi_{\kappa'}(\vec{y})] = i\delta_{\kappa\kappa'}, \quad (3.10)$$

the commutation relation between creation and annihilation operators of scalar field should be defined

$$[a_{\vec{p}}, a_{\vec{p}'}^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{p}'), \quad (3.11)$$

and the energy of particle.

The Hamiltonian of Klein–Gordon theory can be derived from Legendre transformation of Lagrangian (3.1)

$$\hat{H}_{KG} = \int d^3x \frac{\theta_v}{2}\left(g^{tt'}\partial_{t'}\phi\partial_{t'}\phi - g^{qq'}\partial_q\phi\partial_{q'}\phi + m^2\phi^2\right). \quad (3.12)$$

The Hamiltonian of Klein–Gordon theory can be represented by annihilation and creation Operators as

$$\hat{H}_{KG} = \frac{1}{4}\int \frac{d^3p}{(2\pi)^3} p_t\left(a_{\vec{p}}a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}\right). \quad (3.13)$$

The field operator of scalar (3.8) can rotate to interaction picture

$$\phi_I(x) = e^{i\hat{H}_{KG}t} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-i\vec{p}.\vec{x}} + a^\dagger_{\vec{p}} e^{i\vec{p}.\vec{x}}\right) e^{i\hat{H}_{KG}t}, \quad (3.14)$$

and

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-i\vec{p}.\vec{x}} + a^\dagger_{\vec{p}} e^{i\vec{p}.\vec{x}}\right). \quad (3.15)$$

The scalar field as follow is written in interaction picture, and the interaction picture index I is omitted, by default.

Proof: Prove equation (3.10).

The equation (3.10) gives us that

$$\int d^3y \theta_\nu g^{\mu t} [\phi(\vec{x}), \partial_\mu \phi(\vec{y})] = \int d^3y \theta_\nu g^{\mu t} \left([\phi^-(\vec{x}), \partial_\mu \phi^+(\vec{y})] + [\phi^+(\vec{x}), \partial_\mu \phi^-(\vec{y})]\right) \quad (3.16)$$

The label (3.17) should follow the equation (3.17), and the equation (3.17) is

$$\int d^3y \theta_\nu g^{\mu t} [\phi^-(\vec{x}), \partial_\mu \phi^+(\vec{y})] = \int d^3y d^3p \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} \theta_\nu g^{qt} i p'_q [a_{\vec{p}}, a'_{\vec{p}}] - e^{-i(\vec{p}.\vec{x} - \vec{p}'.\vec{y})} = \int d^3y \frac{d^3p}{(2\pi)^3} \frac{i g^{qt} p_q \theta_\nu}{2\omega_{\vec{p}}} e^{-i\vec{p}.(\vec{x}-\vec{y})} = \int d^3y \frac{i g^{qt} p_q \theta_\nu}{2\omega_{\vec{p}}} \delta^3(\vec{x} - \vec{y}) = \frac{i g^{qt} p_q \theta_\nu}{2\omega_{\vec{p}}},$$

(3.17) similarly, we have

$$\int d^3y \theta_\nu g^{\mu t} [\phi^+(\vec{x}), \partial_\mu \phi^-(\vec{y})] = \frac{i g^{qt} p_q \theta_\nu}{2\omega_{\vec{p}}}. \quad (3.18)$$

The rotation

$$\hat{H}_0 = e^{i\hat{H}_0 t} \hat{H}_0 e^{-i\hat{H}_0 t}$$

can reveal the interaction picture of fields. Then for Klein–Gordon theory the commutation relation (3.10) between canonical position and momentum is proved. □
Proof: The Hamiltonian can be written as (3.13).
The Hamiltonian of Klein–Gordon theory is

$$\hat{H}_{KG} = \int d^3x \frac{\theta_\nu}{2} g^{tt'} \partial_t \phi \partial_{t'} \phi - g^{qq'} \partial_q \phi \partial_{q'} \phi + m^2 \phi^2 = \int d^3x \frac{\theta_\nu}{2} [g^{tt'} (\partial_t \phi^- + \partial_t \phi^+)(\partial_{t'} \phi^- + \partial_{t'} \phi^+)] - g^{qq'} (\partial_q \phi^- + \partial_q \phi^+)(\partial_{q'} \phi^- + \partial_{q'} \phi^+) + m^2 (\phi^- + \phi^+)(\phi^- + \phi^+),$$

(3.19) where

$$\partial_q \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{i p_q}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-i\vec{p}.\vec{x}} - a^{\dagger - i\vec{p}.\vec{x}}_{\vec{p}}\right), \quad (3.20)$$

$$\partial_t \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{p^{-ipt}}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-i\vec{p}.\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}.\vec{x}}\right), \quad (3.21)$$

At first, we choose the terms to analyze

$$\int d^3x \frac{\theta_\nu}{2} (g^{tt'} \partial_t \phi^- \partial_{t'} \phi^- - g^{qq'} \partial_q \phi^- \partial_{q'} \phi^- + m^2 \phi^- \phi^-) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p}{2 \cdot (2\pi)^3} \frac{d^3x}{\sqrt{2\omega_{\vec{p}}}} \frac{\theta_\nu}{\sqrt{2\omega_{\vec{p}'}}} \left(-g^{tt'} p_t p_{t'} + g^{qq'} p_q p_{q'} + m^2\right) a_{\vec{p}} a_{\vec{p}'} e^{-i(\vec{p}+\vec{p}').\vec{x}} = = \int \frac{d^3p}{(2\pi)^3} d^3x \frac{\theta_\nu}{2 \cdot 2\omega_{\vec{p}}} \left(-g^{tt} p_t p_t - g^{qq'} p_q p_{q'} + m^2\right) a_{\vec{p}} a_{\vec{p}} = 0. \quad (3.22)$$

Similarly, we have

$$\int d^3x \frac{\theta_\nu}{2} (g^{tt'} \partial_t \phi^+ \partial_{t'} \phi^+ - g^{qq'} \partial_q \phi^+ \partial_{q'} \phi^+ + m^2 \phi^+ \phi^+ = 0. \quad (3.23)$$

And

$$\int d^3x \frac{\theta_\nu}{2} (g^{tt'} \partial_t \phi^- \partial_{t'} \phi^+ - g^{qq'} \partial_q \phi^- \partial_{q'} \phi^+ + m^2 \phi^- \phi^+) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{d^3x}{\sqrt{2\omega_{\vec{p}}}} \frac{\theta_\nu}{2 \cdot \sqrt{2\omega_{\vec{p}}}} \left(g^{tt'} p_t p_{t'} - g^{qq'} p_q p_{q'} + m^2\right) a_{\vec{p}} a^\dagger_{\vec{p}} e^{-i(\vec{p}-\vec{p}).\vec{x}} = \int \frac{d^3p}{(2\pi)^3} d^3x \frac{g^{tt} p_t p_t \theta_\nu}{2 \cdot 2\omega_{\vec{p}}} a_{\vec{p}} a^\dagger_{\vec{p}}, \quad (3.24)$$

then the Klein–Gordon Hamiltonian is

$$\hat{H}_{KG} = \int \frac{d^3p}{(2\pi)^3} d^3x \frac{g^{tt} p_t p_t \theta_\nu}{2.2\omega_{\vec{p}}} \left(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}\right). \quad (3.25)$$

The Hamiltonian also can be written

$$\hat{H}_{KG} = \int d^3x \frac{\theta_\nu}{2} \left(g^{t\mu} \partial_t \phi \partial_\mu \phi - g^{q\mu} \partial_q \phi \partial_\mu \phi + m^2 \phi^2\right), \quad (3.26)$$

and

$$\hat{H}_{KG} = \int \frac{d^3p}{(2\pi)^3} \frac{g^{t\mu} p_t p_\mu \theta_\nu}{2.2\omega_{\vec{p}}} \left(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}\right) = \frac{1}{4} \int d^3p (2\pi)^3 p_t \left(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}\right). \quad \square \ (3.27)$$

Proof: From equation (3.14) prove equation (3.15).
The commutation relation

$$\hat{H}_{KG} a^\dagger_{\vec{p}} = \left(\hat{H}_{KG} + p_t\right) a^\dagger_{\vec{p}}, \quad (3.28)$$

$$\hat{H}^2_{KG} a^\dagger_{\vec{p}} = \left(\hat{H}_{KG} + p_t\right)^2 a^\dagger_{\vec{p}}, \quad (3.29)$$

$$\vdots \quad (3.30)$$

$$\widehat{H}^{n}_{KG} a^{+}_{\vec{p}} = (\widehat{H}_{KG} + p_t)^{n} a^{+}_{\vec{p}}, \quad (3.31)$$

and the Taylor expansion of the unitary transformation

$$e^{-i\widehat{H}_{KG}t} = 1 - i\widehat{H}_{KG}t - \frac{1}{2}\widehat{H}^{2}_{KG}t^{2} - i\frac{1}{3!}\widehat{H}^{3}_{KG}t^{3} + \ldots$$

gives us that

$$a^{\dagger}_{\vec{p}} e^{-i\widehat{H}_{KG}t} = e^{-i\widehat{H}_{KG}t} e^{ip_t t} a^{\dagger}_{\vec{p}}, \quad (3.32)$$

such that

$$e^{-i\widehat{H}_{KG}t} a^{\dagger}_{\vec{p}} e^{-i\widehat{H}_{KG}t} = e^{ip_t t} a^{\dagger}_{\vec{p}}. \quad (3.33)$$

Similarly, we have

$$e^{-i\widehat{H}_{KG}t} a_{\vec{p}} e^{-i\widehat{H}_{KG}t} = e^{-ip_t t} a_{\vec{p}}. \quad (3.34)$$

Then, from equation (3.14) we can derive the scalar field in interaction picture (3.15)

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{\sqrt{2\omega_{\vec{p}}}} a_{\vec{p}} e^{-ip.x} + a^{\dagger}_{\vec{p}} e^{ip.x} \right). \square \quad (3.35)$$

Canonical Quantization of Dirac Theory in Curved Space-time

As another kind example, we choose the Dirac theory in 4-dimensional curved space-time **M** to analyze

$$\hat{L}_{Dirac} = \int d^3x \left[i\bar{\psi}\gamma^{a}(\partial_{\mu}\psi)\theta^{\mu}_{a} - \bar{\psi}m\psi \right]\theta_{\nu}, \quad (4.1)$$

where ψ is Dirac fermions (spin-1/2) in curved space-time. The canonical positions are

$$\phi_k = \{\psi, \bar{\psi}, \theta^{\mu}_{a}\},, \quad (4.2)$$

the corresponding canonical momentum are

$$\pi_k = \frac{\partial \hat{L}_{Dirac}}{\partial \phi_k} = \{i\int d^3x\bar{\psi}\gamma^{a}\theta^{t}_{a}\theta_{\nu}, 0, 0\}. \quad (4.3)$$

The ψ can be expanded by creation and annihilation operator as follow

$$\psi(\vec{x}) = \psi^{-} + \psi^{+} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega\vec{p}}} \sum_{s=1,2} \left(b^{s}_{\vec{p}} u^{s} e^{-i\vec{p}.\vec{x}} + d^{s\dagger}_{\vec{p}} v_{s} e^{i\vec{p}.\vec{x}} \right), \quad (4.4)$$

the anti-commutation relation is defined

$$\left\{ b^{r}_{\vec{p}}, \quad b^{s\dagger}_{\vec{p}'} \right\} = \left\{ d^{r}_{\vec{p}}, \quad d^{s\dagger}_{\vec{p}'} \right\} = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')\delta^{rs}.$$

then the anti-commutation relation of the canonical position and momentum (2.14) for Dirac theory in curved space-time can be derived

$$\{\phi_{\kappa}(\vec{x}), \pi_{\kappa'}(\vec{y})\} = 4i\delta_{\kappa\kappa'}, \quad (4.5)$$

The free Hamiltonian $\widehat{H}_0$ of Dirac theory is derived from Legendre transformation

$$\widehat{H}_{Dirac} = \int d^3x \left(-i\bar{\psi}\gamma^{a}\theta^{q}_{a}\partial_{q}\psi + \bar{\psi}m\psi \right)\theta_{\nu}. \quad (4.6)$$

Similar with Klein-Gordon theory in curved space-time, the free Hamiltonian of Dirac theory in curved space-time can be expressed by creation and annihilation operators

$$\widehat{H}_{Dirac} = \frac{1}{2}\int d^3p \sum_{s} p_t \left(b^{s\dagger}_{\vec{p}} b^{s}_{\vec{p}} + d^{s}_{\vec{p}} d^{s\dagger}_{\vec{p}} \right), \quad (4.7)$$

and the (4.4) can rotate into interaction picture

$$\psi_1(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega\vec{p}}} \sum_{s=1,2} \left(b^{s}_{\vec{p}} u^{s} e^{-ip.x} + d^{s\dagger}_{\vec{p}} v^{s} e^{ip.x} \right). \quad (4.8)$$

Proof: The reason why (4.5) and (4.6) are written should be proved.

The Lagrangian of Dirac theory in curved space-time is

$$\hat{L}_{Dirac} = \int d^3x \left[i\bar{\psi}\gamma^{a}(\partial_{\mu}\psi)\theta^{\mu}_{a} - \bar{\psi}m\psi \right]\theta_{\nu}, \quad (4.9)$$

The Euler-Lagrange equation of Lagrangian (4.9) is the Dirac equation in curved space-time when we choose canonical position $\phi_k = \{\bar{\psi}_i\}$

$$i\gamma^{a}(\partial_{\mu}\psi)\theta^{\mu}_{a} = m\psi \quad (4.10)$$

Substitute the right and left parts of equation (4.4) to equation (4.10) give us that

$$\begin{pmatrix} -m & p.\sigma \\ p.\bar{\sigma} & -m \end{pmatrix} u^{s}(p) = \begin{pmatrix} -m & -p.\sigma \\ -p.\bar{\sigma} & -m \end{pmatrix} v^{s}(p) = 0, \quad (4.11)$$

the solutions of equation (4.11) are

$$u^{s}_{i}(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^{s} \\ \sqrt{p.\bar{\sigma}}\xi^{s} \end{pmatrix}, v^{s}_{i}(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta^{s} \\ -\sqrt{p.\sigma}\eta^{s} \end{pmatrix}, \quad (4.12)$$

where

$$\xi^{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, n^{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, n^{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.13)$$

Proof: Please prove the anti-commutation relation (4.5) of canonical position and momentum of Dirac theory in curved space–time

$$\{\phi_k(\vec{x}),\pi_{k'}(\vec{y})\} = 4i\delta_{kk'}. \quad (4.14)$$

The anti–commutation relation of fermions creation and annihilation operators gives
us that

$$i\int d^3y\theta_a^t\theta_v\left\{\psi(\vec{x}),\bar{\psi}(\vec{y})\gamma^a\right\} = i\int d^3y\theta_a^t\theta_v\left(\left\{\psi^-(\vec{x}),\bar{\psi}^-(\vec{y})\gamma^a\right\}+\left\{\psi^+(\vec{x}),\bar{\psi}^+(\vec{y})\gamma^a\right\}\right). \quad (4.15)$$

We choose the first part to analyze

$$i\int d^3y\theta_a^t\theta_v\left\{\psi^-(\vec{x}),\bar{\psi}^-(\vec{y})\gamma^a\right\} = i\int d^3y\frac{d^3p}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega\vec{p}}}\frac{1}{\sqrt{2\omega\vec{p}}}\sum_{s,\gamma}\theta_a^t\theta_v\left\{b_{\vec{p}}^s u^s, b_{\vec{p}}^{r\dagger}\bar{u}^r\gamma^a\right\}e^{i(\vec{p}.\vec{x}-\vec{p}.\vec{y})} =$$

$$i\int d^3y\frac{d^3p}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega\vec{p}}}\frac{1}{\sqrt{2\omega\vec{p}'}}\sum_{s,\gamma}\theta_a^t\theta_v\left\{b_{\vec{p}}^s, b_{\vec{p}}^{r\dagger}\right\}tr\left(u^s\bar{u}^r\gamma^a\right)e^{i(\vec{p}.\vec{x}-\vec{p}.\vec{y})} =$$

$$i\int d^3y\frac{d^3p}{(2\pi)^3}\frac{\theta_v}{2\omega_{\vec{p}}}\theta_a^t tr\left(\gamma^b p_\mu\theta_b^\mu\gamma^a\right)ei\vec{p}.(\vec{x}-\vec{y}) = 2i.$$

Similarly, we have

$$i\int d^3y\theta_a^t\theta_v\left\{\psi^+(\vec{x}),\bar{\psi}^+(\vec{y})\gamma^a\right\} = 2i,$$

and

$$i\int d^3y\theta_a^t\theta_v\left\{\psi(\vec{x}),\bar{\psi}(\vec{y})\gamma^a\right\} = 4i. \;\square$$

Canonical Quantization of Maxwell Theory in Curved space–time

In Feymann gauge, we deal with the Maxwell Lagrangian terms with gauge fixing condition as follow

$$\hat{L}_{Maxwell} = \int d^3x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2\right]\theta_v, \quad (5.1)$$

where A_i is electro–magnetic 4–vector potential (spin–1) and the gauge strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5.2)$$

The Euler–Lagrange equation for canonical position A_ν of Maxwell Lagrangian is

$$\partial^\mu\partial_\mu A^\nu - \partial^\nu\partial_\mu A^\mu = 0. \quad (5.3)$$

Then the equation is derived

$$\partial^\mu\partial_\mu A^\nu = 0, \quad (5.4)$$

the equation (5.4) tells us that the free photon in curved space–time is mass free

$$p^\mu p_\mu = 0. \quad (5.5)$$

The second quantized canonical positions of quantum electrodynamics in curved space–time are

$$\phi_k = \{A_\mu, \theta_a^\mu\}, \quad (5.6)$$

the corresponding canonical momentums are

$$\pi_k = \{-\int d^3x\partial^t A^\mu\theta_v, 0\}. \quad (5.7)$$

The gauge 4–potential of electro–magnetic field A^μ can be expanded as

$$A^\mu(\vec{x}) = A^{\mu-} + A^{\mu+} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Sigma_{r=1,2}(\varepsilon_{\vec{p}}^{\mu,r}a_{\vec{p},r}e^{-i\vec{p}.\vec{x}} + \varepsilon_{\vec{p}}^{\mu,r*}a_{\vec{p},r}^\dagger e^{i\vec{p}.\vec{x}}), \quad (5.8)$$

where

$$A_\nu = g_{\nu\mu}A^\mu, \quad (5.9)$$

and the commutation relation between creation and annihilation operators of gauge 4–potential A^μ should be defined

$$[a_{\vec{p},r}, a_{\vec{p}',r'}^+] = (2\pi)^3\delta_{rr'}\delta^3(\vec{p}-\vec{p}'), \quad (5.10)$$

The canonical commutation relation between canonical position and momentum of Maxwell theory

$$[\phi_k(\vec{x}),\pi_{k'}(\vec{y})] = 2i\delta_{kk'}, \quad (5.11)$$

gives us that

$$-[A_\mu(\vec{x}),\int d^3y\partial^t A^{\mu+}(\vec{y})]\theta_v = -\int A_\mu^-(\vec{x}),\int d^3y\partial^t A^{\mu+}(\vec{y})]\theta_v - [A_\mu^+(\vec{x}),\int d^3y\partial^t A^{\mu-}(\vec{y})]\theta_v. \quad (5.12)$$

We choose the first term to analyze

$$-[A^-{}_\mu(\vec{x}),\int d^3y\partial^t A^{\mu+}(\vec{y})]\theta_\nu =$$
$$-i\int d^3y\theta_\nu \frac{d^3p}{(2\pi)^3}\frac{p^t}{2\omega_{\vec{p}}} \qquad (5.13)$$
$$\sum{}_r \varepsilon_{\vec{p}}^{\mu,r}\varepsilon_{\vec{p},\mu}^{r*}e^{-i\vec{p}.(\vec{x}-\vec{y})}=i.$$

Then, we have

$$-[A_\mu(\vec{x}),\int d^3y\partial^t A^\mu(\vec{y})]\theta_\nu = 2i. \quad (5.14)$$

The gauge 4-potential can rotate to interaction picture

$$A^\mu(\vec{x})=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\sum{}_{r=1,2}(\varepsilon_{\vec{p}}^{\mu,r}a_{\vec{p},r}e^{-ip.x}+\varepsilon_{\vec{p}}^{\mu,r*}a_{\vec{p},r}^{+}e^{ip.x}).$$

(5.15)

Canonical Quantization of Yang-Mills Theory in Curved Space-time

The Lagrangian of Yang-Mills theory in curved space-time is

$$\hat{L}_{YM}=\int d^3x\left[-\frac{1}{4}F_{\mu\nu}^{\alpha}F^{\mu\nu\alpha}+\bar{\psi}(i\gamma^a\partial_\mu+g\gamma^aA_\mu^\alpha t^\alpha-m)\theta_a^\mu\psi\theta_\nu\right],$$

(6.1)

where the α is gauge group index, t^α are generators of SU(N) group. After the R(gauge fixing term being added

$$-\frac{1}{2\xi}(\partial^\mu A_\mu^\alpha)^2, \quad (6.2)$$

the ghost fields c^α and $c^{-\alpha}$ should be added to cancel the

effects of gauge fixing term, the total Lagrangian of Yang-Mills theory in curved space-time should be BRST (Becchi, Rouet, Stora and Tyutin) invariant [10-12]

$$\hat{L}_{YM-T}=\hat{L}_{YM}+\int d^3x[-\frac{1}{2\xi}(\partial^\mu A_\mu^\alpha)^2+ \qquad (6.3)$$
$$(\partial^\mu c^{-\alpha})\partial_\mu c^\alpha+gf^{\alpha\beta\gamma}(\partial^\mu c^{-\alpha})A_\mu^\beta c^\gamma]\theta_\nu$$

$f^{\alpha\beta\gamma}$ is the structure constant of SU(N) group.

$$F_{\mu\nu}^\alpha=\partial_\mu A_\nu^\alpha-\partial_\nu A\alpha_\mu^\alpha+gf^{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma \quad (6.4)$$

The total Lagrangian of Yang-Mills theory can devide into kinematic and interaction parts

$$\hat{L}_{YM-T}=\hat{L}_0+\hat{L}_{\acute{E}}=(\hat{L}_{Dirac,i}+\hat{L}_{Maxwell}^\alpha)+(\hat{L}_{FP0}^\alpha+\hat{L}_{Gluon}+\hat{L}_{FPI}), \quad (6.5)$$

where $\hat{L}_{Gluon}$ describes the three and four gluon self-interactions in curved space-time, the Lagrangian

$$\hat{L}_{FPI}=g\int d^3xf^{\alpha\beta\gamma}(\partial^\mu c^{-\alpha})A_\mu^\beta c^\gamma\theta_\nu \quad (6.6)$$

describes the ghost-boson-anti-ghost interaction in curved space-time. The canonical quantization of Dirac theory and Maxwell theory in curved space-time are shown, except that Dirac Lagrangian with index i and Maxwell Lagrangian with index α. The free Lagrangian of ghost fields is

$$\hat{L}_{FP0}=\int d^3x(\partial^\mu c^{-\alpha})\partial_\mu c^\alpha\theta_\nu \quad (6.7)$$

Ghost fields are Grassmann number valued, to exchange the adjacent ghost fields a minus symbol should be added. Then the expansion formulation of ghost fields is

$$c^\alpha(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega\vec{p}}}\left(d_{\vec{p}}^\alpha e^{-ip.x}+d_{\vec{p}}^{\alpha\dagger}e^{ip.x}\right), (6.8)$$

$$c^{-\alpha}(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega\vec{p}}}\left(e_{\vec{p}}^\alpha e^{-ip.x}-e_{\vec{p}}^{\alpha\dagger}e^{ip.x}\right). \quad (6.9)$$

The canonical momentum of ghost fields is

$$\int d^3x\theta_\nu\partial^t c^{-\alpha}(x)=\int\frac{d^3xd^3p-ip^t\theta_\nu}{(2\pi)^3\sqrt{2w_{\vec{p}}}}(d_{\vec{p}}^\alpha e^{-ip.x}-d_{\vec{p}}^{\alpha+}e^{ip.x}),$$

$$\int d^3x\theta_\nu\partial^t c^{-\alpha}(x)=\int\frac{d^3xd^3p-ip^t\theta_\nu}{(2\pi)^3\sqrt{2w_{\vec{p}}}}(e_{\vec{p}}^\alpha e^{-ip.x}+e_{\vec{p}}^{\alpha+}e^{ip.x})$$

The anti-commutation relations between canonical positions and momentums are found

$$\{c^\alpha(\vec{x}),\int d^3x\theta_\nu\partial^t c^{-\beta}(\vec{y})\}=i\delta^{\alpha\beta} \quad (6.10)$$

$$\{c^{-\alpha}(\vec{x}),\int d^3x\theta_\nu\partial^t c^{\beta}(\vec{y})\}=-i\delta^{\alpha\beta} \quad (6.11)$$

The free Hamiltonian of ghost fields is

$$\hat{H}_{FPO}=\int d^3x[(\partial^t c^{-\alpha})\partial_t c^\alpha-(\partial^q c^{-\alpha})\partial_q c^\alpha]= \qquad (6.12)$$
$$\int\frac{d^3p}{(2\pi)^3}p_t(e_{\vec{p}}^\alpha d_{\vec{p}}^{\alpha\dagger}-e_{\vec{p}}^{\alpha\dagger}d_p^\alpha).$$

There are commutation relations

$$[\hat{H}_{FP0},e_{\vec{p}}^\alpha]=-p_te_{\vec{p}}^\alpha, \quad (6.13)$$

$$[\hat{H}_{FP0},d_{\vec{p}}^\alpha]=-p_td_{\vec{p}}^\alpha, \quad (6.14)$$

and

$$[\hat{H}_{FP0},e_{\vec{p}}^{\alpha\dagger}]=p_te_{\vec{p}}^{\alpha\dagger}, \quad (6.15)$$

$$[\hat{H}_{FP0},d_{\vec{p}}^{\alpha\dagger}]=p_td_{\vec{p}}^{\alpha\dagger}. \quad (6.16)$$

The equations (6.13), (6.14), (6.15) and (6.16) derive that

$$e^{i\bar{H}_0 t} e^{\alpha}_{\vec{p}} e^{-i\bar{H}_0 t} = e^{\alpha}_{\vec{p}} e^{-ip^t t},$$

$$e^{i\bar{H}_0 t} e^{\alpha\dagger}_{\vec{p}} e^{-i\hat{H}_0 t} = e^{\alpha}_{\vec{p}} e^{ip^t t},$$

$$e^{i\bar{H}_0 t} d^{\alpha}_{\vec{p}} e^{-i\bar{H}_0 t} = d^{\alpha}_{\vec{p}} e^{-ip^t t},$$

$$e^{i\hat{H}_0 t} d^{\alpha\dagger}_{\vec{p}} e^{-i\hat{H}_0 t} = d^{\alpha}_{\vec{p}} e^{-ip^t t},$$

Discussions and Conclusion

Yang-Mills theory is the theoretic framework of Standard Model (SM) of particle physics with gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)_L\times\mathrm{U}(1)$. The general relativity is geometry theory of gravity with curved space-time. The second quantized Yang-Mills theory with classical gravity theory gives us the quantum field theory in curved space-time. The canonical quantization of Yang-Mills theory in curved space-time has interesting theoretical meaning and maybe experimental insights. The canonical quantization of Klein–Gordon, Dirac, Maxwell and Yang–Mills theories based on Lagrangian and Hamiltonian in local coordinates gives us self–consistent canonical quantization framework. From local analysis to the global analysis of quantum field theory is interesting. The behaviors of quantum field behind the black hole event horizon and singularity are theoretically important to probe the self–consistency of quantum field in curved space–time. The relations between quantum field theory, simplistic geometry and quantum statistic should be researched further.

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