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Evolution and Dynamics of Quantum Fluids

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Abstract

Quantum Fluids follow Quantum Dynamical Equation(s), which were not known till date. There exist a set of two equations that is semi classical approach to Quantum Fluids called Madelung's Equations. But a new fully quantum variant of Madelung's Equations when embedded in the Schrodinger Equation is gives full description of evolution of Quantum fluid with respect to time and position. The equation presented in this article has two unknown variables, one is density and other is velocity field as a function of spatial and time coordinates. The equation presented in this article, is derived from Schrodinger Equation, obeying Continuity equation, and Navier Strokes Equation. Bohm's potential was externally added in Madeline's equation. But the new equation which is fully quantum mechanical in nature; Bohm's potential appears out of the equation, which is interesting to observe. Astrophysical cold stellar dynamics and condensed fluids have the main application of this equation. Quantum fluids show strange behaviour when compared to normal fluids. It is also shown that quantum fluid also has spins which has no classical analog.

Keywords: Quantum Fluids; Fluid dynamic; Superfluid

Introduction

Quantum hydrodynamics is one of the new research fields in fluid dynamics. Low-temperature superfluid's show quantum mechanical behaviour. Classical hydrodynamics is governed by a single equation known as Navier-Strokes equation [1]. Often Navier Strokes equation is solved either computationally or using certain assumptions and boundary conditions. A complete analytical problem of the Navier-Strokes equation is still a challenge and also a Millennium problem. Recent developments in Mathematics, showed that finite time blowup of 3 dimensional Navier-Strokes equation exists using averaged methods. Existence of exact solution in 3 dimension is what some people are working on. Then a question may arise that do quantum fluids obey the Navier strokes equation [2]. The answer is yes. For quantum fluids, there exists a set of two equations called Madelung's Equations [3], which is a quantum variant of Navier Strokes equation and continuity equation for classical fluids. Madelung's Equations are semi-classical equations that tell about the dynamics of Quantum Fluids. But "is this enough ?, or we need a full quantum equation for defining the dynamics of quantum fluids.

Examples of Quantum Fluids include the presence of a large number of neutrons in a comparatively small volume in neutron stars. They deviate from Maxwell- Boltzmann statistics and follow Fermi-Dirac Statistics, which is one part of Quantum Statistical Mechanics. Another common example is Bose-Einstein's Condensates that do not follow Maxwell Boltzmann Statistics [4]. To explain these large varieties of fluids that are important in condensed matter physics and astrophysics requires a new equation that will explain the dynamics of these systems.

This article shows that such an equation exists that is fully quantum in nature and is derived from the Navier-Strokes equation and continuity equation embedded in

Schrodinger Equation. This new equation is fully quantum because it is Schrodinger's Equation variant fluid equation and it must describe the dynamics of Quantum fluids as it comes from classical continuity and Navier Strokes equation. A rigorous study of this equation can open new ideas of research, as it is believed that Cold Stars like white dwarfs and neutrons stars are composed of Quantum fluids. Also, Helium-4 is proven to follow Quantum phenomena. Another interesting argument is incorporation of spin in fluids, which has no analogy in Classical Fluids.

Some important concepts used to work on this project involves ideas of multiparticle eigenfunction because this was obviously unavoidable. Fluids are treated as continuum media and hence it is convenient to talk about field. In classical case, it is velocity field which is a type of vector field. The new equation starts with describing multiparticle wave function and then observing how this idea of discrete multiparticle can be systematically mapped to cumulative behaviour.

Describing the Eigen-Function of Quantum Fluids

Taking a brief look at II will tell that the number density of any Quantum Fluids can be analogously related to the mod square of the quantum mechanical wave function. Hence the normalization constant that appears in Schrodinger equation [5] is no more a constant but square root of number density which is a function of spatial coordinates and time. This idea can be fetched on philosophical grounds of Copenhagen interpretation. Born Rule says that for a single particle

$$\iiint_v |\psi|^2 d\Omega = P(V)$$

P(V) is probability of finding a particle within volume V. Suppose the whole system is actually confined in the given volume, then P is unity. Therefore,

$$\iiint_v |\psi|^2 , d\Omega \approx |\psi|^2 V = 1$$

$$|\psi|^2 \approx \frac{1}{V}$$

Define multiparticle eigenfunction as $\psi = \Pi_{i=1}^{N} \otimes \phi_{\mu}\left(x^{\mu}, t\right)$ where ψ is defined on $\mathbb{R}^{3N} \times \mathbb{R}^{+}$ (N is number of particles). The above formulation will help us on building the work further. It is mentioned in abstract section that, to solve the dynamics of Quantum fluid we need two fields, one is velocity field and other one is density field. So it is important to connect the velocity field with some analogous variable in the Schrodinger equation. It appears in the calculation in II that the phase of the quantum mechanical wave function is actually related to the velocity field.

Therefore, the analogous parameter to eigenfunction is Schrondinger Equation for Quantum fluids can be written as $\sqrt{\rho\left(x^{\mu}, t\right)} exp\left(i\kappa u\left(x^{\mu}, t\right) x_{\mu}\right)$ where x^{μ} denotes the spatial coordinates for $\mu = 1, 2, 3$. And κ is a constant.

Revisiting Madelung's Equations

Madelung's Equation is a set of two-equation derived from the classical continuity equation for fluids and the Navier Strokes Equation. Madelung's second equation also incorporates Bohm's potential given by Q. The physical significance of Bohm's potential is properly described in David Bohm's article on the interpretation of Quantum Mechanics (famously called Bohmian Mechanics or pilot-wave theory).

$$\frac{\partial \rho_m}{\partial t} + \hat{\nabla}.\left(\rho_m \hat{u}\right) = 0 \qquad (1)$$

$$\frac{\partial \rho_m}{\partial t} + \hat{u}.\hat{\nabla}\hat{u} = \frac{-1}{m}\nabla(Q + \hat{V}) \qquad (2)$$

Where ρ_m is number density and it is interpreted as $|\psi|^2$.

The detailed analogy of this is given in subsection IA and the mathematical description is in II. This approach is clearly a semi classical approach as Schrodinger equation is not applied rather the interpretation of Schrodinger equation is taken into account.

How Single Particle Behaviour Can be Traced as Navier Strokes Equation

The very famous time dependent Schrodinger equation, that defines non-relativistic Quantum mechanics for many body system is written as

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{8\pi^2 m}\sum_{i=1}^{N} \nabla^2 \psi + \hat{V}\psi \qquad (3)$$

Where Ψ is multiparticle wave function defined as product of individual wave functions. Hydrodynamic Equation of Continuity is written as

$$\frac{\partial \rho}{\partial t} + \nabla\left(\rho u\right) = 0 \qquad (4)$$

The complex conjugate of ψ is $\psi\dagger$. So, Equation (3) for $\psi\dagger$

instead of ψ is

$$i\hbar\frac{\partial|\psi|^2}{dt}=-\frac{\hbar^2}{8\pi^2 m}\sum_{i=1}^{N}\nabla^2\psi+\hat{V}\psi\dagger \quad (5)$$

If we multiply Equation (3) with $\psi\dagger$ and Equation (5) with ψ and subtracting Equation (5) from Equation (3)

$$i\hbar\frac{\partial|\psi|^2}{dt}=-\frac{\hbar^2}{8\pi^2 m}\sum_{i=1}^{N}\left(\psi\dagger\nabla_i^2\psi-\psi\nabla_i^2\psi\dagger\right)$$

$$-\frac{\hbar^2}{8\pi^2 m}\sum_{i=1}^{N}\nabla_i\left(\psi\dagger\nabla\psi-\psi\nabla_i\psi\dagger\right) \quad (6)$$

$$\frac{\partial|\psi|^2}{dt}=i\frac{\hbar}{8\pi^2 m}\sum_{i=1}^{N}\nabla_i\left(\psi\dagger\nabla_i\psi-\psi\nabla_i\psi\dagger\right) \quad (7)$$

See that equation (7) is rearrangement of Equation (6), such that Equation (6) becomes comparable with Equation (4).But according to Max Born's interpretation of Quantum Mechanics: Volume integral of $|\psi|^2$ is probability. But if $|\psi|^2$ = ρ, then Volume integral of $|\psi|^2$ cannot be probability, as dimensions of right and left sides do not match. To fix the problem of dimensional analysis, we have to divide Equation (4) with the mass of particles, of which the fluid is composed, Note that, if Equation (4) is divided with mass of the particles involved, then it matches with Madelung's First Equation(i.e., Equation (1)). It must be noted that in continuity equation only one partial differential with respect to spatial co-ordinate(s) appear(s), but here there is a sum of partial differentials with respect to the coordinates of individual particles. This can be solved if we take a small infinitesimal volume element with is composed of N particles and the differentials with respect to individual coordinates can be transformed to centre of mass coordinate. So,

$$x_{com}=\frac{\sum_{i=1}^{N}x_i}{N}$$

so, trivially,

$$\nabla_i=\frac{1}{N}\nabla_{com}$$

so the sum of all the partial derivate equal partial derivate with respect to COM coordinate. Therefore, $|\psi|^2$ = $\frac{\rho}{m}$. Now the interesting part of comparing Equation (4) with Equation (7), is the comparison of the second term in both equations.

$$u=\frac{-i\hbar}{8\pi^2\mathrm{m}|\psi|^2}\sum_{i-1}^{N}\left(\dagger\nabla_i\psi-\psi\nabla_i\psi\dagger\right)$$

$$=\frac{-i\hbar}{8\pi^2\mathrm{m}}\left(\frac{\nabla_{com}\psi}{\psi}-\frac{\nabla_{com}\psi\dagger}{\psi\dagger}\right)$$

$$=\frac{-i\hbar}{8\pi^2\mathrm{m}}\nabla_{com}ln\left(\frac{\psi}{\psi\dagger}\right) \quad (8)$$

The wave function in Quantum mechanics is usually written as $\psi = Aexp\left(ikx-i\omega t\right)$. Therefore, Equation (8) changes to

$$\int u.dr=\frac{=i\hbar}{8\pi^2\mathrm{m}}2\left(ikx-i\omega t\right) \quad (9)$$

Writing $\int u.dr\approx u.r.$ Therefore wave function in Quantum Mechanics can to fluid dynamics variable is

$$\psi\rightarrow\sqrt{\frac{\rho}{m}}exp\left(\frac{4i\pi^2 m}{\hbar}u.r\right) \quad (10)$$

Schrodinger-Navier Strokes Equation for Quantum Fluids

Refer to Equation (3). Partial derivative with respect to time must Change to material derivative or Laplace Derivative such that

$$\frac{\partial}{\partial t}\rightarrow\frac{D}{Dt}=\frac{\partial}{\partial t}+u.\nabla \quad (11)$$

So the new form of Equation (3) contains material derivate instead of partial derivate.

$$i\hbar\frac{\partial\psi}{\partial t}+i\hbar u.\nabla\psi=-\frac{\hbar^2}{8\pi^2 m}\nabla^2\psi+\hat{V}\psi \quad (12)$$

The potential V contain the potential from Navier Stroke's Equation as well as Bohm's potential. Quantum fluids are believed to be in viscid fluids. Therefore, their kinetic viscosity is zero. Putting all this conditions Navier Stroke's equation looks like

$$\rho\left(\frac{\partial u}{\partial t}+\left(u.\nabla\right)u\right)=-\nabla P\left(x^{\mu},t\right)+\rho\hat{g} \quad (13)$$

The left side of Equation (13) is the force per unit volume due to kinetic Energy and the right side denotes the for force per unit volume due to potential energy term. I will simply call them kinetic and potential force for simplicity.

Force can be written as negative gradient of potential energy. Now,

$$\frac{F}{V} = -\nabla P + \rho\hat{g}$$

$$= -\frac{\nabla\hat{V}}{V} \quad (14)$$

$\hat{V}$ denote Potential Energy and V is for Volume. This just chosen to remove the confusion due to notation. Right hand sides of Equation must be equal.

$$-\nabla\hat{V} = \left(-\nabla P + \nabla\int\rho\hat{g}.dr\right)V \quad (15)$$

Put Equation (15) on Equation (12)

$$i\hbar\frac{\partial\psi}{\partial t} + i\hbar u.\nabla\psi = -\frac{\hbar^2}{8\pi^2 m}\nabla^2\psi + P\left(x^\mu, t\right)V\psi - \left(V\int\rho\hat{g}.dr\right)\psi \quad (16)$$

I can put the final expression, replacing ψ in equation (16) with Equation (10). This will be a very simple but tedious calculation. Therefore, it is useful to write the final expression directly.

$$i\hbar\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t} - \left(4\pi^2 m\right)\sqrt{\rho r}.\frac{\partial u}{\partial t} + i\hbar\frac{u.\nabla\rho}{2\sqrt{\rho}} -$$

$$\left(4\pi^2 m\right)\sqrt{\rho}\left(u.\nabla\right)\left(u.r\right) = -\frac{\hbar^2}{16\pi^2 m}\frac{\nabla^2\rho}{\sqrt{\rho}} + \frac{\hbar^2}{32\pi^2 m}\frac{\left(\nabla\rho\right)^2}{\rho^{3/2}}$$

$$+\frac{\hbar^2}{8\pi m}\frac{\nabla\rho}{\sqrt{\rho}}\nabla\left(u.r\right) + \frac{\hbar^2}{8\pi^2 m} + \frac{i\hbar}{2}\sqrt{\rho}\left[\left(\nabla\left(u.r\right)\right)^2 + \nabla^2\left(u.r\right)\right]$$

$$+PV\sqrt{\rho} - \left(V\int\rho\hat{g}.dr\right)\sqrt{\rho}$$

The final equation can be broken into two part: one that contain iota as a factor and second is the real part. Equation (16) does not include the spin as in the absence of magnetic field spin remains degenerate. But system with local or global magnetic field like neutron stars, Pauli's Equation [6] has to be used. This implies that quantum fluids have intrinsic spin, which has no classical analog. This article does not go that further but the point remains interesting and opens a new phenomenon for fluids. What spin of a fluid mean probably has no explanation, but it does mean that some different and strange behaviour can be observed in the case of Quantum Fluids. For spin 1/2 fluids there must be two different densities.

A pure Quantum equation has been proposed taking Madelung's equation into consideration along with Schrodinger Equation. The equation is proposed to govern the dynamics of quantum fluids. This has to be done by correlating Wave function continuity equation and Continuity equation. Calculations showed that in case of quantum fluids the wave function can be replaced by $\sqrt{\frac{\rho}{m}exp\left(\frac{4i\pi^2 m}{\hbar}u.r\right)}$. Bohm's potential automatically, appear in the equation, therefore it has not been used externally, only Navier Strokes potential terms have been added.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

1. Potter M, Wiggert DC (2008) Fluid Mechanics. McGraw-Hill.
2. Tao T (2015) Finite time blowup for an averaged three dimensional navier strokes equation. arXiv, 1402.0290.
3. Madelung E (1926) Eine anschauliche deutung der gleichung von schrödinger. Naturwissenschaften 14: 1004.
4. Ballagh R, Savage C (2000) Bose-Einstein condensation: from atomic physics to quantum fluids. In: Savage C, Das M (Eds.), Proceedings of the 13th Physics Summer School, World Scientific, Singapore.
5. Schrodinger E (1926) An Undulatory Theory of the Mechanics of Atoms and Molecules. Phys Rev 28: 1049-1070.
6. Pauli W (1927) On the quantum mechanics of the magnetic electron. Magazine for physics 43: 601–623.

