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Pati-Salam Model in Curved Space-Time from Square Root Lorentz Manifold

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Abstract

There is a $U(4')\times U(4)$ -bundle on four-dimensional square root Lorentz manifold. Then a Pati-Salam model in curved space-time (Lagrangian) and a gravity theory (Lagrangian) are constructed on square root Lorentz manifold based on self-parallel transportation principle. An explicit formulation of Sheaf quantization on this square root Lorentz manifold is shown. Sheaf quantization is based on superposition principle and construct a linear Sheaf space in curved space-time. The transition amplitude in path integral quantization is given which is consistent with Sheaf quantization. All particles and fields in Standard Model (SM) of particle physics and Einstein gravity are found in square root metric and the connections of bundle. The interactions between particles/fields are described by Lagrangian explicitly. There are few new physics in this model. The gravity theory is Einstein-Cartan kind with torsion. There are new particles, right handed neutrinos, dark photon, Fiona, $X^{\pm c}$ and $Y^0, Y^1, Y^2, Y^1_*, Y^2_*$.

Keywords: Lorentz manifold; Pati-Salam model; Curved space-time; Gravity theory; Yang-Mills theory; Sheaf quantization

Introduction

Four-dimensional pseudo-Riemann geometry with signature $(-,+,+,+)$, Lorentz manifold, is the geometry background of the general relativity, space-time is described by the metric, and the gravitational field is described as the curve of space-time. In general relativity, the geodesic equation describes the trajectories of free particles, and the Einstein equation determines how matter curves space-time. At the last life time of Einstein, he attempted to establish a new geometry unifying electromagnetic interaction and gravity. This idea was developed by Weil into the early idea of gauge invariance and by the Kaluza and Klein into the idea of extra dimensions.

Later, the Yang-Mills theory [1] was confirmed. Yang-Mills theory takes gauge invariance as its basic principle and to be the theoretical framework of electromagnetic, weak and strong interaction in Standard Model (SM) of particle physics [2-7]. The Yang-Mills theory is the theoretical framework of SM and has a good correspondence with the complex structure group G fiber bundle theory [8]. General relativity can actually be rewritten in the framework of fiber bundle theory also, except that the G structure group of general relativity is real, and the corresponding fiber bundles are the tangent and cotangent bundles. Another way to build unified field theory is introducing extra dimensions to give all fields their geometric positions. And lots of attempts in extra dimension were made. Is it possible to fuse the tangent (cotangent) bundle of general relativity with the complex structure group G-bundle of Yang-Mills theory?

Inspired by the Dirac's way of finding his equation and spinors through making square root of the Klein-Gordon equation, we researched the four-dimensional square root Lorentz manifold, which similar with the papers in Clifford algebra or Clifford bundle [9-52], spin-gauge theory in Riemann-Cartan space-time [53,54], sedenion [55] and Einstein-Cartan theory [56-58] etc. Four-dimensional square root Lorentz manifold has extra $U(4') \times U(4)$ principal bundle than Lorentz manifold. Two Lagrangians based on four-dimension square root Lorentz manifold are constructed which describe a $U(4') \times U(4)_L \times U(4)_R$ Pati-Salam model in curved space-time and a gravity theory, respectively. In the Pati-Salam model [59], the $SU(4')$ is color group with "lepton number as the fourth color", and the $SU(4)_L \times SU(4)_R$ is chiral flavor group. The gravity theory is Einstein kind with torsion. We realize an explicit formulation of Sheaf quantization [60-76] scheme which consistent with path integral quantization. The particles spectrum on this model is analyzed.

Geometry and Lagrangian

The notations are introduced here. a,b,c,d represent frame indices and a,b,c,d are equal to $0,1,2,3$. μ,ν,ρ,σ represent coordinates indices and μ,ν,ρ,σ are equal to $0,1,2,3$. α represent group indices with α equals to $0,1,....,15$. i,j,k,l,m are equal to $1,2,3,4$. C is quarks color and equals to $R,G,B(1,2,3)$. k is Sheaf space index. Repeated indices are summed by default.

The pseudo-Riemann manifold is described by a metric

$$g(x) = -g_{\mu\nu}(x)dx^\mu dx^\nu, \quad (1)$$

where the metric is symmetric

$$g_{\mu\nu}(x) = g_{\nu\mu}(x), \quad (2)$$

and $\{x \mid x = (x^\mu) = (t,\vec{x})\}$ is a four-dimensional topological space. Here we discuss the four-dimensional pseudo-Riemann manifold with signature $(-,+,+,+)$, Lorentz manifold. And it can be described by orthonormal frame (vierbein) formalism as

$$g^{-1}(x) = -\eta^{ab}\theta_a(x)\theta_b(x), \quad (3)$$

where

$$\eta^{ab} = diag(1,-1,-1,-1).$$

orthonormal frames describe gravitational field (Figure 1).

$$\theta_a(x) = \theta_a^\mu(x)\frac{\partial}{\partial x^\mu}.$$

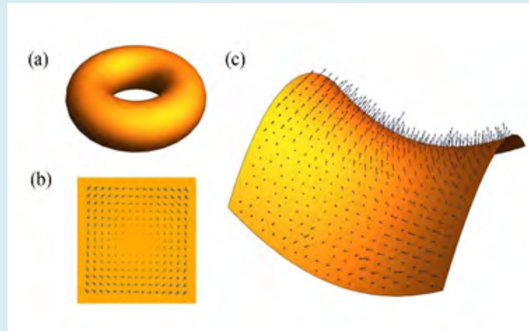

Figure 1: Picture visualization of geometry background of (a) general relativity (b) Yang-Mills theory and (c) square root metric. (a) Geometry background of general relativity is pseudo-Riemann geometry, which is a smooth curved manifold. (b) Geometry background of Yang-Mills theory is complex G-bundle with flat base manifold, G is the gauge group of Yang-Mills theory. (c) Square root metric geometry in four-dimension has everything of pseudo-Riemann geometry and extra $U(4') \times U(4)$ -bundle.

The definition of gamma matrices is

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I_{4\times4}. \quad (4)$$

The Hermiticity conditions for gamma matrices are

$$\gamma^a\gamma^{b\dagger} + \gamma^{b\dagger}\gamma^a = 2I^{ab}I_{4\times4}, \quad (5)$$

where I^{ab} is $diag(1,1,1,1)$. We define

$$l = i\gamma_{ik}^0(x)\gamma_{kj}^a(x)e_j^\dagger \otimes e_j\theta_a(x), \quad (6)$$

$$\tilde{l} = i\gamma_{ik}^a(x)\gamma_{kj}^0(x)e_j^\dagger \otimes e_i\theta_a(x), \quad (7)$$

where e_i are the orthogonal bases expanding four-dimension complex space $\mathbb{C}^4$. The orthogonal bases of $\mathbb{C}^4$ satisfy

$$tr(e_j^\dagger \otimes e_i) = e_ie_j^\dagger = \delta_{ij}. \quad (8)$$

One simple choice of e_i is

$$e_1 = (e^{i\theta_1},0,0,0), e_2 = (0,e^{i\theta_2},0,0), \quad (9)$$

$$e_3 = (0,0,e^{i\theta_3},0), e_4 = (0,0,0,e^{i\theta_4}). \quad (10)$$

After using $\gamma^{a\dagger} = \gamma^0\gamma^a\gamma^0$, we find that

$$g^{-1}(x)=\frac{1}{4}tr\left[\tilde{l}(x)l(x)\right]. \quad (11)$$

Then $l(x)$ and $\tilde{l}(x)$ are the square root of metrics in some sense. The representation freedom of $\gamma^{a}_{ij}(x)$ can be shown as

$$\gamma^{0'}_{ik}(x)\gamma^{a'}_{kj}(x)=\psi^{\dagger}_{ik}(x)\gamma^{0}_{kl}\gamma^{a}_{lm}\psi_{mj}(x)=\bar{\psi}_i(x)\gamma^{a}\psi_j(x),$$

$$\gamma^{a'}_{ik}(x)\gamma^{0'}_{kj}(x)=\psi^{\dagger}_{ik}(x)\gamma^{a}_{kl}\gamma^{0}_{lm}\psi_{mj}(x)=\bar{\psi}_i(x)\gamma^{a\dagger}\psi_j(x),$$

where ψ_i are the Dirac fermions field with flavor related index *i* equals to 1,2,3,4. And

$$\bar{\psi}_i(x)=\psi^{\dagger}_i(x)\gamma^{0},\psi(x)\in U(4)$$

is 4×4 matrix. So, the square root metrics are defined as follow

$$l(x)=i\bar{\psi}_i(x)\gamma^{a}\psi_j(x)e^{\dagger}_j\otimes e_i\theta_a(x), \quad (12)$$

$$\tilde{l}(x)=i\bar{\psi}_i(x)\gamma^{a\dagger}\psi_j(x)e^{\dagger}_j\otimes e_i\theta_a(x). \quad (13)$$

The square root Lorentz manifold is described by square root metric (12), (13). Direct calculations show that the definition (12) and (13) satisfy (11) and

$$l^{\dagger}(x)=-l(x),\tilde{l}^{\dagger}(x)=-\tilde{l}(x).$$

The coefficients of the affine connections on coordinates, coefficients of spin connections on orthonormal frame [77] and gauge fields on the $U(4')\times U(4)$-bundle are defined as follows

$$\nabla_{\mu}\partial_{\nu}=\Gamma^{\rho}_{\nu\mu}(x)\partial_{\rho}, \quad (14)$$

$$\nabla_{\mu}\theta_a(x)=\Gamma^{b}_{a\mu}(x)\theta_b(x), \quad (15)$$

$$\nabla_{\mu}\left(\gamma^{0}\gamma^{a}\right)=i\left[V_{\mu}(x)\gamma^{0}\gamma^{a}-\gamma^{0}\gamma^{a}V_{\mu}(x)\right], \quad (16)$$

$$\nabla_{\mu}e_i=iW_{\mu ij}(x)e^{\dagger}_j. \quad (17)$$

A relation between coefficients of the affine connections on coordinates and coefficients of spin connections on orthonormal frame can be easily found

$$\Gamma^{b}_{a\mu}(x)\theta^{\rho}_b(x)=\partial_{\mu}\theta^{\rho}_a(x)+\theta^{\nu}_a(x)\Gamma^{\rho}_{\nu\mu}(x).$$

If the covariant derivative is compatible with metric,

$$\nabla g(x)=0,$$

then

$$\Gamma_{ab\mu}=-\Gamma_{ba\mu}.$$

The gauge fields on the $U(4')\times U(4)$-bundle are Hermitian

$$V^{\dagger}_{\mu}(x)=V_{\mu}(x),W^{*}_{\mu ij}(x)=W_{\mu ji}(x).$$

The uniqueness of definition of gauge fields is originated from restriction (4), (5) and (8). The equation as follow can be derived from (16)

$$\nabla_{\mu}\left(\gamma^{a}\gamma^{0}\right)=i\left[\tilde{V}_{\mu}(x)\gamma^{a}\gamma^{0}-\gamma^{a}\gamma^{0}\tilde{V}_{\mu}(x)\right], \quad (18)$$

where we write

$$\tilde{V}_{\mu}(x)=\gamma^{0}V\gamma^{0}.$$

The gauge field $V_{\mu}(x)$ and $W_{\mu ij}(x)$ can be decomposed by the generators of the $U(4)$ group

$$V_{\mu}(x)=V^{\alpha}_{\mu}(x)\mathcal{T}^{\alpha}, \quad (19)$$

$$W_{\mu ij}(x)=W^{\alpha}_{\mu}(x)\mathcal{T}^{\alpha}_{ij}, \quad (20)$$

where α is equals to $0,1,2,\dots.15$. There are Hermitian gauge bosons fields

$$V^{\alpha\dagger}_{\mu}(x)=V^{\alpha}_{\mu}(x),W^{\alpha\dagger}_{\mu}(x)=W^{\alpha}_{\mu}(x).$$

The $\mathcal{T}^{\alpha}$ are the generators of $U(4)$ and an expilict one can be seen in appendix. An equation is constructed which satisfying the $U(4')\times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant principles

$$\boldsymbol{tr}\nabla\left[l(x)\right]=0. \quad (21)$$

This equation originated from generalized self-parallel transportation principle. Eliminating index *x*, the explicit formula of equation (21) is

$$[(i\partial_{\mu}\bar{\psi}_i-\bar{\psi}_i\tilde{V}_{\mu}+W_{\mu ij}\bar{\psi}_j)\gamma^{a}\psi_i+\bar{\psi}_i\gamma^{a}$$
$$(i\partial_{\mu}\psi_i+V_{\mu}\psi_i-\psi_jW_{\mu ji})+i\bar{\psi}_i\gamma^{b}\psi_i\Gamma^{a}{}_{b\mu}]\theta_a=0$$

We define a Lagrangian

$$\mathcal{L}=\bar{\psi}_i\gamma^{a}\left(i\partial_{\mu}\psi_i+V_{\mu}\psi_i-\psi_jW_{\mu ji}\right)\theta^{\mu}_a+\frac{i}{2}\bar{\psi}_i\gamma^{b}\psi_i\Gamma^{a}{}_{b\mu}\theta^{\mu}_a. \quad (22)$$

The last term in Lagrangian (22) is Yukawa coupling term $\psi_i\phi\psi_i$ and the scalar (Higgs) field is gamma matrix valued and originated from gravitational field.

$$\phi=\frac{i}{2}\gamma^{b}\Gamma^{a}{}_{b\mu}\theta^{\mu}_a. \quad (23)$$

Then, the Lagrangian (22) describes $U(4')\times U(4)$ Yang-Mills theory in curved space-time (Figure 1). The Lagrangian (22) has relation with (21)

$$\boldsymbol{tr}\nabla\left[l(x)\right]=\mathcal{L}-\mathcal{L}^{\dagger}. \quad (24)$$

If equation (21) being satisfied, the Lagrangian (22) is Hermitian

$$\mathcal{L}=\mathcal{L}^{\dagger}. \quad (25)$$

So, the unitary principle of quantum field theory (25) consistent with generalized self-parallel transportation principle (21). The equations of motion for the Lagrangian (22) are

$$\gamma^{a}\left(i\partial_{\mu}\psi_{i}+V_{\mu}\psi_{i}-\psi_{j}W_{\mu ji}\right)\theta_{a}^{\mu}+\frac{i}{2}\gamma^{b}\psi_{i}\Gamma^{a}{}_{b\mu}\theta_{a}^{\mu}=0, \quad (26)$$

and this equation's conjugate transpose. We point out that (21) is density matrix version of (26). The effective equation of motion of (26) has signature (1, −1, −1, −1). For example, the massless Dirac equation in curved space-time of this model is

$$i\gamma^{a}\theta_{a}^{\mu}\partial_{\mu}\psi_{i}=0. \quad (27)$$

The square of equation (27) is massless Klein-Gordon equation in curved space-time

$$\eta^{ab}\theta_{a}^{\mu}\theta_{b}^{\nu}\partial_{\mu}\partial_{\nu}\psi_{i}=0. \quad (28)$$

The signature of equation (28) is $(1,-1,-1,-1)$ and consistent with special relativity. Then, a Lagrangian (22) which describes the $U(4')\times U(4)$ Yang-Mills theory in curved space-time is all other fields (except Higgs field) dynamical background which satisfies the characteristic of the gravitational field in our real world.

Lagrangian (22) is $U(4')\times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant. So, Lagrangian (22) is demanded invariant under the transformations

$$\begin{aligned}&\psi_{i}^{'}=\tilde{U}\psi_{j}U_{ji},\\&\gamma^{a'}=\tilde{U}\gamma^{b}\tilde{U}^{\dagger}\Lambda_{b}{}^{a}, \quad (29)\\&\theta_{a}^{'\mu}=\Lambda_{a}{}^{b}\theta_{b}^{\mu},\end{aligned}$$

Then, the transformation rules have to be derived as follows

$$V_{\mu}^{'}=\tilde{U}V_{\mu}\tilde{U}^{\dagger}-\left(\partial_{\mu}\tilde{U}\right)\tilde{U}^{\dagger}, \quad (30)$$

$$V_{\mu}^{'}=U_{ki}^{*}W_{\mu}U_{lj}+U_{ki}^{*}\left(\partial_{\mu}U_{kj}\right), \quad (31)$$

$$\Gamma^{'b}_{a\mu}=\Lambda_{a}^{c}\Gamma^{d}{}_{c\mu}\Lambda_{d}^{b}-\Lambda_{a}{}^{c}\partial_{\mu}\Lambda_{c}{}^{b}, \quad (32)$$

where the transformation matrices satisfy

$$\tilde{U}\tilde{U}^{\dagger}=I, U_{ji}U_{jk}^{*}=\delta_{ik}, \Lambda_{a}{}^{b}\Lambda_{b}{}^{c}=\delta_{a}^{c}.$$

Then the transformation elements

$$\tilde{U}\in U(4'),$$

$$\left(U_{ij}\right)\in U(4),$$

$$\Lambda_{b}{}^{a}\in O(1,3),$$

and $U(4')$ is color group, $U(4)$ is flavor group, $O(1,3)$ is locally Lorentz group. Then we complete the proof of the Lagrangian (22) is $U(4')\times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant.

The gauge field strength tensors and curvature tensor are defined as follows

$$H_{\mu\nu}=\partial_{\mu}V_{\nu}-\partial_{\nu}V_{\mu}-iV_{\mu}V_{\nu}+iV_{\nu}V_{\mu},$$

$$F_{\mu\nu ij}=\partial_{\mu}W_{\nu ij}-\partial_{\nu}W_{\mu ij}-iW_{\mu ik}W_{\nu kj}+iW_{\nu ik}W_{\mu kj},$$

$$R^{a}{}_{b\mu\nu}=\partial_{\mu}\Gamma^{a}{}_{b\nu}-\partial_{\nu}\Gamma^{a}{}_{b\mu}+\Gamma^{c}{}_{b\nu}\Gamma^{a}{}_{c\mu}-\Gamma^{c}{}_{b\mu}\Gamma^{a}{}_{c\nu}.$$

If the covariant derivative is compatible with metric, the curvature tensor satisfies

$$R_{ab\mu\nu}=-R_{ba\mu\nu}.$$

The gauge fields strengths on the $U(4')\times U(4)$-bundle match the Hermiticity conditions

$$H_{\mu\nu}^{\dagger}=H_{\mu\nu}, F_{\mu\nu ij}^{*}=F_{\mu\nu ji}.$$

The gauge field strength can be decomposed by the $U(4)$ generators

$$H_{\mu\nu}=H_{\mu\nu}^{\alpha}\mathcal{T}^{\alpha}, F_{\mu\nu ij}=F_{\mu\nu}^{\alpha}\mathcal{T}_{ij}^{\alpha}.$$

After the torsion being defined

$$T_{\nu\rho}^{a}=2\Gamma^{a}{}_{[\nu\rho]}, \quad (33)$$

we have the Ricci identity and Bianchi identity [78] on this geometry structure as follows

$$\partial_{[\mu}H_{\nu\rho]}=H_{[\mu\nu}V_{\rho]}-V_{[\mu}H_{\nu\rho]}, \quad (34)$$

$$\partial_{[\mu}F_{\nu\rho]ij}=F_{[\mu\nu|ik|}W_{\rho]kj}-W_{[\mu|ik|}F_{\nu\rho]kj}, \quad (35)$$

$$T^{a}{}_{\sigma[\rho}T_{\mu\nu]}=R^{a}_{[\rho\mu\nu]}+\nabla_{[\rho}T_{a\mu\nu]}, \quad (36)$$

$$\nabla_{[\rho}R^{a}_{\;|b|\mu\nu]} = R^{a}_{b\sigma[\rho}T_{\mu\nu]}. \quad (37)$$

There is Yang-Mills Lagrangian for gauge bosons in this model

$$\mathcal{L}_{\Upsilon} = \frac{-1}{2}\boldsymbol{tr}\left(H^{\mu\nu}H_{\mu\nu}\right) - \frac{-\zeta}{2}F^{\mu\nu}_{ij}F_{\mu\nu ji}, \quad (38)$$

where $\zeta \in R$ is constant.

For the gravity, the Einstein-Hilbert action in Lorentz manifold be showed as follow

$$S = \int \omega R, \quad (39)$$

where is the Ricci scalar curvature in Lorentz manifold, ω is volume form

$$\omega = \sqrt{-g_v}\,dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

and

$$g_v = det\left(g_{\mu\nu}(x)\right).$$

In this geometry framework, the equations can be derived as follows

$$\nabla_{[\mu}\nabla_{\nu]}l = \frac{-1}{2}(\psi_i\gamma^a\psi_k F_{\mu\nu kj} - F^{*}_{\mu\nu ki}\overline{\psi}_k\gamma^a\psi_j + \overline{\psi}_i\tilde{H}_{\mu\nu}\gamma^a\psi_j$$

$$-\overline{\psi}_i\gamma^a H_{\mu\nu}\psi_j + \frac{i}{2}\overline{\psi}_i\gamma^b\psi_j R^{a}_{\;b\mu\nu})e^{\dagger}_j \otimes e_i\theta^a, \quad (40)$$

$$\nabla_{[\mu}\nabla_{\nu]}\tilde{l} = \frac{-1}{2}(\psi_i\gamma^{a\dagger}\psi_k F_{\mu\nu kj} - F^{*}_{\mu\nu ki}\overline{\psi}_k\gamma^{a\dagger}\psi_j + \overline{\psi}_i\tilde{H}_{\mu\nu}\gamma^{a\dagger}\psi_j$$

$$-\overline{\psi}_i\gamma^{a\dagger}H_{\mu\nu}\psi_j + \frac{i}{2}\overline{\psi}_i\gamma^{b\dagger}\psi_j R^{a}_{\;b\mu\nu})e^{\dagger}_j \otimes e_i\theta^a, \quad (41)$$

where we write

$$\tilde{H}_{\mu\nu} = \gamma^0 H_{\mu\nu}\gamma^0.$$

We define

$$\nabla^2 = \nabla_{[\mu}\nabla_{\nu]}dx^\mu \wedge dx^\nu,$$

the equation of a gravity theory is constructed

$$\boldsymbol{tr}\nabla^2\left[\tilde{l}(x)l(x)\right] = 0. \quad (42)$$

This equation (42) is obviously $U(4')\times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant. The explicit formula of equation (42) is

$$R\psi^{\dagger}_i\psi_i = i(F^{abij}\psi^{\dagger}_j\left(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger}\right)\psi_i$$

$$-\psi^{\dagger}_i H_{ab}\left(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger}\right)\psi_i, \quad (43)$$

where we use the notations as follow,

$$\partial_\mu dx^\nu \otimes dx^\rho \partial_\sigma = \delta^\nu_\mu \delta^\rho_\sigma,$$

$$dx^\mu \otimes dx^\nu \partial_\rho \partial_\sigma = \delta^\nu_\rho \delta^\mu_\sigma,$$

$$F_{abij} = F_{abij}\theta^\mu_a\theta^\nu_b,$$

$$H_{ab} = H_{\mu\nu}\theta^\mu_a\theta^\nu_b.$$

So, we define a $U(4')\times U(4)$ gauge invariant, locally Lorentz invariant, generally covariant Lagrangian

$$\mathcal{L}_g = R\psi^{\dagger}_i\psi_i - i(F^{abij}\psi^{\dagger}_j\left(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger}\right)\psi_i$$
$$+i\psi^{\dagger}_i H_{ab}\left(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger}\right)\psi_i. \quad (44)$$

The Lagrangian (44) is Hermitian

$$\mathcal{L}_g = \mathcal{L}^{\dagger}_g. \quad (45)$$

The $R\psi^{\dagger}_i\psi_i$ in Lagrangian (44) gives us the Einstein-Hilbert action. The equation (43) and the Einstein tensor can be derived from the Lagrangian (44). The gravity theory in this model is Einstein-Cartan kind with torsion.

Sheaf Quantization and Path Integral Quantization

The entities $l(x)$ and $\tilde{l}(x)$ are two sections of the two bundles, respectively, where these two bundles are dual to each other. Further, the Sheaf valued entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$

are superposition of sections $l(x)$ and $\tilde{l}(x)$ (Figure 2).

$$\hat{l}(x) = \sum_\kappa \eta_\kappa(x)|\kappa, x\rangle\langle\kappa, x|l_\kappa(x), \quad (46)$$

$$\hat{\tilde{l}}(x) = \sum_\kappa \eta_\kappa(x)|\kappa, x\rangle\langle\kappa, x|\tilde{l}_\kappa(x), \quad (47)$$

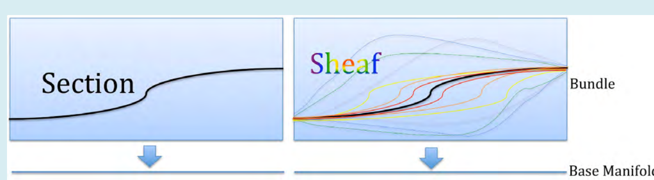

Figure 2: The fiber bundle structure is shown. Left: l(x) is a section of the bundle. Right: A Sheaf is a collection of the sections. $\hat{l}(x)$ is Sheaf valued.

where $\eta_\kappa(x) \in [0,1]$ are probability of corresponding section $l_\kappa(x)$ and $\tilde{l}_\kappa(x)$, κ is Sheaf space index and evaluated in an abelian group. The density matrix corresponds to $\hat{l}(x)$ and $\tilde{l}(x)$ is

$$\rho(x) = \sum_\kappa \eta_\kappa(x) |\kappa, x\rangle\langle\kappa, x|. \quad (48)$$

We have orthogonal bases in Sheaf space. The orthogonal bases in Sheaf space satisfy probability complete formulas

$$\langle\kappa, x \mid \kappa', x'\rangle = \delta(x - x')\delta(\kappa - \kappa'). \quad (49)$$

$$\boldsymbol{tr}\,\rho(x) = \sum_{\hat{\kappa}} \eta_\kappa(x) = 1. \quad (50)$$

In mathematics, a Sheaf is a collection of sections, the index κ of each section correspond to an abelian group element. In physics, the Sheaf spaces $Sh(x)$ and $\widetilde{Sh}(x)$ are expanded by all possible sections $l(x)$ and $\tilde{l}(x)$ of the two bundles, respectively. The $Sh(x)$ and $\widetilde{Sh}(x)$ are linear spaces, which means, for example, any two entities $\hat{l}_1(x)$ and $\hat{l}_2(x)$ in $Sh(x)$, there is an entity $\hat{l}(x)$ in $Sh(x)$ equals to the mixing of the two entities

$$\begin{aligned} &\hat{l}(x) = \eta_1(x)\hat{l}_1(x) + \eta_2(x)\hat{l}_2(x), \\ &\hat{l}_1(x), \hat{l}_2(x) \in Sh(x); \qquad (51) \\ &\Rightarrow \hat{l}(x) \in Sh(x), \end{aligned}$$

where the probability of each section

$$\eta_1(x), \eta_2(x) \in [0,1]$$

and

$$\eta_1(x) + \eta_2(x) = 1.$$

The Sheaf spaces $Sh(x)$ and $\widetilde{Sh}(x)$ are dual to each other. We call it Sheaf quantization which switching study objects from single section to all possible sections of the bundle. The equation (50) derives to equation of motion for density matrix

$$d\left(\boldsymbol{tr}\,\rho(x)\right) = \boldsymbol{tr}\left(d\rho(x)\right) = 0, \quad (52)$$

where d is exterior differential derivative. The equations of motion for entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$ in Sheaf quantization method are

$$\boldsymbol{tr}\nabla\left[\hat{l}(x)\right] = 0, \boldsymbol{tr}\left[\hat{\tilde{l}}(x)\hat{l}(x)\right] = 0. \quad (53)$$

The corresponding total Lagrangian density is

$$\hat{\mathcal{L}} = \sum_\kappa \mathcal{L}_\kappa + g\mathcal{L}_{g,\kappa} + \tilde{g}\mathcal{L}_{Y,\kappa}, \quad (54)$$

where g and $\tilde{g}$ are Lagrangian multipliers with $g, \tilde{g} \in \mathbb{R}$.

For pure state

$$\rho(x) = \left|\Psi(x)\Psi(x)\right|, \quad (55)$$

$$\left|\Psi(x)\right\rangle = \sum_\kappa \alpha_\kappa(x)\left|\kappa\right\rangle, \quad (56)$$

where the $\left|\Psi(x)\right\rangle$ is the quantum state of quantum field theory. The transition amplitude can be defined through path integral formula

$$\alpha_\kappa(t, \vec{x}) = \int_{t' \in (t_0, t)} D\kappa(t', \vec{x}) e^{i\omega\hat{\mathcal{L}}[\kappa(t,\vec{x})]} \alpha_\kappa(t_0, \vec{x}), \quad (57)$$

where ω is volume form.

Particles Spectrum

V_μ^α and W_μ^α (α is equals to $0, 1, \cdots, 15$ here) are gauge bosons fields. The interactions related with the gauge bosons fields W_μ^α always preserves the possibility of chiral symmetry breaking in the Lagrangian (54) such that the gauge group can be decomposed to $U(4') \times U(4)_L \times U(4)_R$, where $U(4')$ is color group and $U(4)_L \times U(4)_R$ is chiral flavor group. The V_μ^0 is dark photon and W_μ^0 is Fiona particle. V_μ^0 and W_μ^0 are related with $U(1')$ and $U(1)$ gauge group in $U(4')$ and $U(4)$, respectively. The left-over gauge group is a Pati-Salam gauge group $SU(4') \times SU(4)_L \times SU(4)_R$ (Figure 3) [59,79]. The $SU(4')$ gauge group can be decomposed as follow

$$U(4') = SU(3') \oplus U(1') + U_{X^+} + U_{X^-}. \quad (58)$$

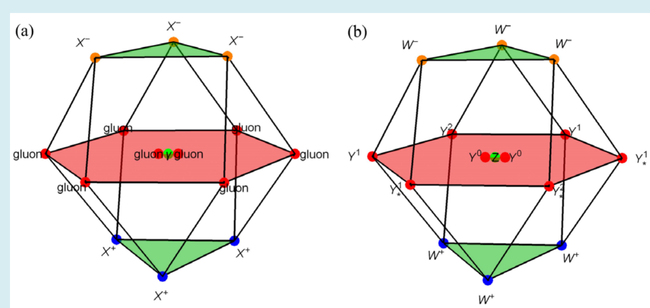

Figure 3: Weight diagram of $SU(4)$ adjoint representation and corresponding gauge bosons. The decomposition of $SU(4)$ adjoint representation is $\mathbf{15} = \mathbf{8} \otimes \mathbf{1} + \mathbf{3} + \mathbf{3}^*$. (a) The wight diagram of V_μ^α which contained gauge bosons are gluons, photon and $X^{\pm C}$. (b) The wight diagram of W_μ^α related gauge bosons are $Y^0, Y^1, Y^2, Y_*^1, Y_*^2$ and $W^\pm, Z$.

The $SU(3')$ is the gauge group of quantum chramodynamics (QCD) and the corresponding gauge bosons V_μ^α (α is equals to $1, 2 \cdots, 8$ here) are gluons. The $U(1')$ in equation (58) is electro-magnetic interaction gauge group and corresponding gauge boson V_μ^{15} is photon γ. The $X^{\pm C}$ particles transport semi-leptonic processes and

$$X^{\pm C}=V_{\mu}^{8+C}\pm iV_{\mu}^{9+C}. \quad (59)$$

The electric charges of each X^{+C} and X^{-C} are 1/3 and −1/3, respectively. The chiral gauge group $SU(4)_{L,R}$ can be decomposed as (Figure 4).

$$SU(4)_{L,R}=SU(3)_{Y}\oplus U(1)_{Z}+U_{W^{+}}+U_{W^{-}}, \quad (60)$$

and related gauge bosons V_{μ}^{α} (with α equals to $1,2,\cdots,15$) contain weak bosons $W^{\pm}$ and Z

$$\begin{aligned}W_{\mu}^{\pm}&=W_{\mu}^{9}\pm iW_{\mu}^{10}=W_{\mu}^{11}\pm iW_{\mu}^{12} \quad (61)\\&=W_{\mu}^{13}\pm iW_{\mu}^{14},\end{aligned}$$

$$Z_{\mu}=W_{\mu}^{15}. \quad (62)$$

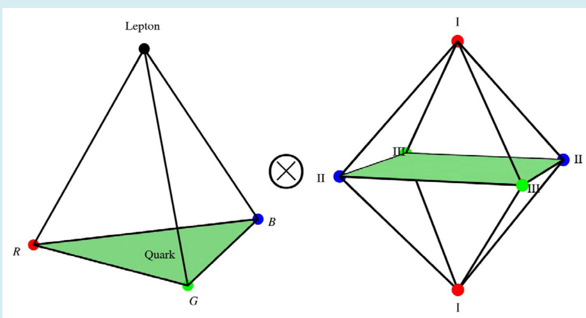

Figure 4: Weight diagram of $SU(4)$ fundamental representation $\mathbf{4}\otimes\mathbf{6}$. Representation $\mathbf{4}=\mathbf{3}+\mathbf{1}$, **3** is 3 kinds of color, red, green and blue, **1** is the lepton number. Representation **6** gives us **6** flavors of quarks and leptons. **6** flavors have 3 generations, I, II and III, each generation has 2 kinds of quarks or leptons.

The left over gauge bosons are $Y^{0},Y^{1},Y^{2},Y_{*}^{1},Y_{*}^{2}$ with 0 electric charge. The gauge bosons $Y^{1},Y^{2},Y_{*}^{1},Y_{*}^{2}$ transport non-SM flavor changing neutral currents (FCNCs) and

$$\begin{pmatrix}(2-\eta)Y_{\mu}^{0} & Y_{\mu}^{1} & Y_{\mu}^{2}\\ Y_{*\mu}^{1} & \eta Y_{\mu}^{0} & Y_{\mu}^{1}\\ Y_{*\mu}^{2} & Y_{*\mu}^{1} & -2Y_{\mu}^{0}\end{pmatrix}=2\sum_{\alpha=1}^{8}W_{\mu}^{\alpha}\mathcal{T}^{\alpha}. \quad (63)$$

The $X^{\pm C}$ and $Y^{1},Y^{2},Y_{*}^{1},Y_{*}^{2}$ must be super heavy from the restrictions of experimental data.

The fermionic fields ψ_i transfer as the $U(4')\times U(4)$ fundamental representation according to (29). So, fermions are filled into the $SU(4)$ fundamental representation $\mathbf{4}\otimes\mathbf{6}$ naturally. The fundamental representation **4** corresponds to **3** colors and **1** lepton and leads us reobtain "Lepton number as the fourth color" [59]. The fundamental representation **6** corresponds to **6** flavors of quarks and leptons. The weight diagram coordinates in Chevalley basis of representation **6** (see in Table 1) have good correspondence with the quark quantum number [80]. The antifermions be filled into the representation $\mathbf{6}\otimes\mathbf{4}$ similarly. The weight diagram of fermions is shown in Figure 4. Both left-handed and right-handed fermions for all quarks and leptons are existed. Especially, the existence of right-handed neutrinos is predicted. This is compatible with experimental results [81] and the well know Seesaw mechanism [82-86]. The $2I_z$ being used in Table I are not no reasons because we have the Gell-Mann-Nishijima formula [80].

$$Q=I_{z}+\frac{\mathfrak{P}+S+C+B+T}{2}. \quad (64)$$

An explicit fermions representation [79] in this model might be

$$\psi_i=\begin{pmatrix}\sqrt{2}u_R & \sqrt{2}c_R & \sqrt{2}t_R & d_R'\\ \sqrt{2}u_G & \sqrt{2}c_G & \sqrt{2}t_G & d_G'\\ \sqrt{2}u_B & \sqrt{2}c_B & \sqrt{2}t_B & d_B'\\ e & \mu & \tau & \nu'\end{pmatrix}, \quad (65)$$

where u,c,t and d' are quarks fields, e,μ,τ and ν' are electron, mu, tau and neutrinos fields. The corresponding fermions electric charges of (65) are

$$Q=\begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ 1 & 1 & 1 & 0\end{pmatrix}. \quad (66)$$

The quarks states like $|d\rangle,|s\rangle,|b\rangle$ and neutrinos states $|\nu_e\rangle,|\nu_u\rangle,|\nu_\tau\rangle$ are eigen states of the Lagrangian [79].

	2Iz	S+C	B+T	H1	H2	H3
u	1	0	0	1	-1	1
d	-1	0	0	-1	1	-1
c	0	1	0	1	0	-1
s	0	-1	0	-1	0	1
t	0	0	1	0	1	0
b	0	0	-1	0	-1	0

Table 1: Corresponding relations between quarks quantum number and weight diagram coordinates in Chevalley basis H1, H2 and H3 of $SU(4)$ representation **6** are shown.

Conclusion and Discussion

A Pati-Salam model and a gravity theory from square root Lorentz manifold are derived. A Sheaf quantization scheme which consistent with path integral quantization is shown. The particles spectrum in this model is discussed.

Some possible new physics on this model are listed as follows. The gravity theory has torsion. There are exotic gauge bosons such as dark photon, Fiona, $X^{\pm C}$ and $Y^0, Y^1, Y^2, Y_*^1, Y_*^2$. The $X^{\pm C}$ transports semi-leptonic processes, the Y^1, Y^2, Y_*^1, Y_*^2 transport non-SM FCNCs. The right-handed neutrinos are existed. The Higgs field is gamma matrix valued.

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