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Square Root Metric Geometry and Pati-Salam Model in Curved Space-Time

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Abstract

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Keywords: Lorentz Manifold; Geometrization Principle of Physics; Pati-Salam Model; Hermitian Property; Lagrangian Density; Sheaf Quantization and Path Integral Quantization

Abstract

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Introduction

It's a very pleasure to have the opportunity to talk about the theory of square root metric geometry and unified field theory for everyone. This physics theory is based on the geometrization principle of fundamental physics, and geometry is a set of logical systems. The Einstein-like school holds that any physical quantity is only understood when it is geometrically presented. A typical example is general relativity. The geometric background of general relativity is smooth, curved Riemannian manifolds, which are described by metric. In general relativity, the gravitational field is geometrically described as the curvature of spacetime. Matter tells space-time how to curve, and space-time tells matter how to move.

The geometric background of the Standard Model of particle physics (SM) is the G fiber bundle structure, where G is the gauge group of the Yang-Mills theory [2]. In the SM, the gauge group is $SU(3')\times SU(2)_L\times U(1)$, and $SU(3')$ is the color gauge group of quantum chromodynamics, describing strong interactions [3,4]. $SU(2)_L\times U(1)$ is the chiral gauge group of the electroweak unified theory, describing the chiral weak interaction and the electromagnetic interaction [5-7]. The SM encompasses all observed fundamental particles, including electrons and photons, as well as u and d quarks that make up protons and neutrons. Protons are composed of uud quarks, and neutrons are composed of udd quarks. Square root metric geometry is a good option for the spatiotemporal context of unified field theory. The square root metric geometry not only has a metric structure but also a G fiber bundle structure. For the 4-dimensional spacetime, there is $U(4')\times U(4)$ gauge group on square root Lorentz manifold [8], which is related to the well-known Yang-Mills existence and mass gap problem. The Pati-Salam model of curved spacetime derived from the self-parallel transportation principle can be regarded as a candidate theory for the unified theory.

Another motivation is to take the square root something usual lead to something unusual, for example, one take the square root of -1 to get the imaginary number i; take the square root of the Klein-Gordon equation to obtain the Dirac equation. What kind of geometry will be obtained by taking the square root of a metric? Let's begin the journey of square root metric geometry.

Square Root Metric Geometry and Pati-Salam Model in Curved Space-Time

The Riemannian manifold can be described by a metric

$$g(x) = -g_{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}, \tag{1}$$

or given by the inverse metric

$$g^{(-1)}(x) = -\eta^{ab}\theta_a(x)\theta_b(x), \tag{2}$$

where

$$\eta^{ab} = diag(1,-1,-1,-1),$$

orthonormal frames describe gravitational field

$$\theta_a(x) = \theta_a^{\mu}(x)\frac{\partial}{\partial x^{\mu}}.$$

It can be seen that the inverse of the metric is related to $l(x)$ and $\tilde{l}(x)$

$$g^{-1}(x) = \frac{1}{4}tr\left[\tilde{l}(x)l(x)\right].$$

The explicit mathematical formulas of $l(x)$ and $\tilde{l}(x)$ are shown in formula

$$l(x) = i\gamma_{ik}^{0}(x)\gamma_{kj}^{a}(x)e_j^{\dagger} \otimes e_i\theta_a(x), \tag{3}$$

$$\tilde{l}(x) = i\gamma_{ik}^{a}(x)\gamma_{kj}^{0}(x)e_j^{\dagger} \otimes e_i\theta_a(x), \tag{4}$$

$l(x)$ and $\tilde{l}(x)$ form a pair of square root metrics. Since the Dirac matrix has degrees of freedom for phase rotation, the square root metric can be written as

$$l(x) = i\bar{\psi}_i(x)\gamma^{a}\psi_j(x)e_j^{\dagger} \otimes e_i\theta_a(x), \tag{5}$$

$$\tilde{l}(x) = i\bar{\psi}_i(x)\gamma^{a\dagger}\psi_j(x)e_j^{\dagger} \otimes e_i\theta_a(x). \tag{6}$$

This pair of square root metrics is anti-Hermitian

$$l^{\dagger}(x) = -l(x), \tilde{l}^{\dagger}(x) = -\tilde{l}(x).$$

The connection of this geometric is defined

$$\nabla_{\mu}\partial_{\nu} = \Gamma_{\nu\mu}^{\rho}(x)\partial_{\rho}, \tag{7}$$

$$\nabla_{\mu}\theta_a(x) = \Gamma_{a\mu}^{b}(x)\theta_b(x), \tag{8}$$

$$\nabla_{\mu}(\gamma^0\gamma^a) = i[V_{\mu}(x)\gamma^0\gamma^a - \gamma^0\gamma^a V_{\mu}(x)], \tag{9}$$

$$\nabla_{\mu}e_i = iW_{(\mu ij)}(x)e_j^{\dagger}. \tag{10}$$

where, eq.(7) is the coefficients of the affine connections on coordinates, eq.(8) coefficients of spin connections on orthonormal frame [9,10], and eq.(9) is gauge field on the U(4')-bundle, eq.(10) is gauge field on the U(4)-bundle.

Based on the principle of self-parallel transportation of semi-metric geometry, the equation of motion can be constructed.

$$tr\nabla\left[l(x)\right] = 0. \tag{11}$$

$$tr\nabla^2\left[\tilde{l}(x)l(x)\right] = 0. \tag{12}$$

This pair of equations of motion is generalized covariant, local frame rotation invariant, and U(4')×U(4) gauge invariant. The relationship between the Lagrangian density and the equation of motion can be given

$$tr\nabla\left[l(x)\right] = \mathcal{L} - \mathcal{L}^{\dagger}. \tag{13}$$

When this geometry satisfies the principle of self-parallel transportation eq.(11), the Lagrangian density is Hermitian.

$$\mathcal{L} = \mathcal{L}^{\dagger}.$$

The Hermitian property of the Lagrangian density ensures the unitary nature of the theory, and the Hermitian property of the Lagrangian density originated from self-parallel transportation principle.

The corresponding effective Lagrangian density for Pati-Salam model in curved space-time is

$$\begin{aligned}\mathcal{L}_{effective} &= \bar{\psi}_i\gamma^a\left(i\partial_{\mu}\psi_i + V_{\mu}\psi_i - \psi_j W_{\mu ji}\right)\theta_a^{\mu} \\ &+ \bar{\psi}_i\phi\psi_i + V(\phi) - \frac{1}{2}tr\left(H^{\mu\nu}H_{\mu\nu}\right) - \frac{\zeta}{2}F_{ji}^{\mu\nu}F_{\mu\nu ij},\end{aligned} \tag{15}$$

before this Lagrangian density is the Dirac kinetic energy term, followed by the minimum coupling term, then the Yukawa coupling term, and then the Higgs potential and the Yang-Mills term. Where $H_{(\mu\nu)}$ and $F_{(\mu\nu ij)}$ are gauge field strength tensors

$$H_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu} - iV_{\mu}V_{\nu} + iV_{\nu}V_{\mu},$$

$$F_{\mu\nu ij} = \partial_{\mu}W_{\mu ij} - iW_{\mu i\kappa}W_{\nu\kappa j} + iW_{\nu i\kappa}W_{\mu\kappa j}.$$

This Lagrangian density eq.(15) describes the Pati-Salam model in curved space-time, and the canonical quantization of this model can be done.

Sheaf Quantization and Path Integral Quantization

The Sheaf space is collection of the sections of a bundle. As $l(x), \tilde{l}(x)$ are sections of the bundles, respectively. The Sheaf valued section can be written

$$\hat{l}(x)=\sum_k \eta_k(x)|k,x\rangle\langle k,x|l_k(x), \tag{16}$$

$$\hat{\tilde{l}}=\sum_\kappa \eta_\kappa(x)|\kappa,x\rangle\langle \kappa,x|\tilde{l}_\kappa(x), \tag{17}$$

where κ is sheaf space index and evaluated in an abelian group. The density matrix

$$\rho(x)=\sum_k \eta_k(x)|k,x\rangle\langle k,x|, \tag{18}$$

is added before a pair of entity, square root metric $l_k(x),\tilde{l}_k(x)$, to embody superposition principle. The superposition principle at sheaf quantization is, if any two entities $\hat{l}_1(x)$ and $\hat{l}_2(x)$ in $Sh(x)$, there is an entity $\hat{l}(x)$ in $Sh(x)$ equals to the mixing of the two entities

$$\begin{aligned}&\hat{l}(x)=\eta_1(x)\hat{l}_1(x)+\eta_2(x)\hat{l}_2(x),\\&\hat{l}_1(x),\hat{l}_2(x)\in Sh(x);\\&\Rightarrow \hat{l}(x)\in Sh(x),\end{aligned} \tag{19}$$

where the probability of each section

$$\eta_1(x),\eta_2(x)\in[0,1]$$

and

$$\eta_1(x)+\eta_2(x)=1.$$

The $Sh(x)$ and $\widetilde{Sh}(x)$ are linear spaces of $l_k(x)$ and $\hat{l}_k(x)$, respectively; and the Sheaf spaces $Sh(x)$ and $\widetilde{Sh}(x)$ are dual to each other. Sheaf quantization switching study objects from single section to all possible sections of the bundle. Sheaf quantization could improve the problem of infinite of quantum field theory. The equations of motion for sheaf valued entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$ in Sheaf quantization method are

$$tr\nabla\left[\hat{l}(x)\right]=0, tr\left[\hat{\tilde{l}}(x)\hat{l}(x)\right]=0. \tag{20}$$

When we have quantum state of quantum field theory $|\psi(x)\rangle$, the pure state of density operator is

$$\rho(x)=|\psi(x)\rangle\langle\psi(x)| \tag{21}$$

is shown. As the quantum state spanned by the bases of sheaf space

$$|\psi(x)\rangle=\sum_k \alpha_k(x)|k\rangle \tag{22}$$

the transition amplitude of path integral formulation can be derived [11]

$$\begin{aligned}&\alpha_\kappa(t,\vec{x})\\&=\int_{t'\in(t_0,t)} D\kappa(t',\vec{x})e^{i\omega\tilde{\mathcal{L}}[\kappa(t,\vec{x})]}\alpha_\kappa(t_0,\vec{x})\end{aligned}$$

Summary

Square root metric geometry has extra U(4')×U(4) bundle, and Pati-Salam model in curved space-time can be constructed by self-parallel transportation principle. This theory is self-consistent under sheaf quantization, path integral quantization, and canonical quantization.

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